Newton, Sir Isaac

# Univerſal Arithmetick:

OR, A

# TREATISE

OF

# ARITHMETICAL Compoſition and Reſolution.

To which is added,

Dr. HALLEY's Method of finding the Roots of Æquations Arithmetically.

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*Tranſlated from the* LATIN *by the late Mr.* RAPHSON, *and reviſed and corrected by Mr.* CUNN.

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# TO THE READER.

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*To ſay any Thing in Praiſe of the enſuing Treatiſe, were an Attempt as needleſs and impertinent, as to write a Panegyrick on its Author. 'Tis enough that the Subject is* Algebra; *and that it was written by Sir* Iſaac Newton: *Thoſe who know any Thing of the Sciences, need not to be told the Value of the former; nor thoſe who have heard any Thing of Philoſophy and Mathematicks, to be inſtructed in the Praiſes of the latter. If any Thing could add to the Eſteem every Body has for the* Analytick *Art, it muſt be, that Sir* Iſaac *has condeſcended to handle it; nor could any Thing add to the Opinion the World has of that illuſtrious*

*illuſtrious Author's Merit, but that he has written with ſo much Succeſs on that wonderful Subject.*

*'Tis true, we have already a great many Books of* Algebra*, and one might even furniſh a moderate Library purely with Authors on that Subject: But as no Body will imagine that Sir* Iſaac *would have taken the Pains to compoſe a new one, had he not found all the old ones defective; ſo, it will be eaſily allow'd, that none was more able than he, either to diſcover the Errors and Defects in other Books, or to ſupply and rectify them in his own.*

*The Book was originally writ for the private Uſe of the Gentlemen of* Cambridge*, and was deliver'd in Lectures, at the publick Schools, by the Author, then* Lucaſian *Profeſſor in that Univerſity. Thus, not being immediately intended for the Preſs, the Author had not proſecuted his Subject ſo far as might otherwiſe have been expected; nor indeed did he ever find Leiſure to bring his Work to a Concluſion: So that it muſt be obſerv'd, that all the Conſtructions, both Geometrical and Mechanical, which occur towards*

*towards the End of the Book, do only ſerve for finding the firſt two or three Figures of Roots; the Author having here only given us the Conſtruction of Cubick Æquations, tho' he had a Deſign to have added, a general Method of conſtructing Biquadratick, and other higher Powers, and to have particularly ſhown in what Manner the other Figures of Roots were to be extracted. In this unfiniſh'd State it continu'd till the Year 1707, when Mr.* Whiſton, *the Author's Succeſſor in the* Lucaſian *Chair, conſidering that it was but ſmall in Bulk, and yet ample in Matter, not too much crowded with Rules and Precepts, and yet well furniſh'd with choice Examples, (ſerving not only as* Praxes *on the Rules, but as Inſtances of the great Uſefulneſs of the Art itſelf; and, in ſhort, every Way qualify'd to conduct the young Student from his firſt ſetting out on this Study) thought it Pity ſo noble and uſeful a Work ſhould be doom'd to a College-Confinement, and obtain'd Leave to make it Publick. And in order to ſupply what the Author had left undone, ſubjoyn'd the General and truly Noble Method of extracting the Roots of Æquations, publiſh'd by Dr.*

Dr. Halley *in the* Philoſophical Tranſactions, *having firſt procur'd both thoſe Gentlemen's Leave for his ſo doing.*

*As to the publiſhing a Tranſlation of this Book, the Editor is of Opinion, that 'tis enough to excuſe his Undertaking, that ſuch Great Men were concern'd in the Original; and is perſwaded, that the ſame Reaſon which engag'd Sir* Iſaac *to write, and Mr.* Whiſton *to publiſh the* Latin *Edition, will bear him out in publiſhing this* Engliſh *one: Nor will the Reader require any farther Evidence, that the Tranſlator has done Juſtice to the Original, after I have aſſur'd him, that Mr.* Raphſon *and Mr.* Cunn *were both concern'd in this Tranſlation.*

# Univerſal Arithmetick;

OR, A

# TREATISE

OF

# Arithmetical COMPOSITION and RESOLUTION.

COMPUTATION is either perform'd by Numbers, as in Vulgar Arithmetick, or by Species, as uſual among Algebraiſts. They are both built on the ſame Foundations, and aim at the ſame End, *viz.* *Arithmetick* Definitely and Particularly, *Algebra* Indefinitely and Univerſally; ſo that almoſt all Expreſſions that are found out by this Computation, and particularly Concluſions, may be call'd *Theorems.* But *Algebra* is particularly excellent in this, that whereas in *Arithmetick* Queſtions are only reſolv'd by proceeding from given Quantities to the Quantities ſought, *Algebra* proceeds, in a retrograde Order, from the Quantities ſought as if they were given, to the Quantities given as if they were ſought, to the End that we may ſome Way or other come to a Concluſion or Æquation, from which one may bring out the Quantity ſought. And after this Way the moſt difficult Problems are reſolv'd, the Reſolutions whereof would be ſought in vain from only common Arithmetick. Yet *Arithmetick* in all its Operations is ſo ſubſervient to *Algebra*, as that they ſeem both but to make one perfect Science of Computing; and therefore I will explain them both together.

Whoever goes upon this Science, muſt firſt underſtand the Signification of the Terms and Notes, [or Signs] and learn the fundamental Operations, *viz. Addition, Subſtraction, Multiplication*, and *Diviſion*; *Extraction of Roots, Reduction of Fractions*, and *Radical Quantities*; and the *Methods of ordering the Terms of Æquations*, and *exterminating the unknown Quantities*, (where they are more than one). Then let [the Learner] proceed to exerciſe [or put in Practice] theſe Operations, by bringing Problems to Æquations; and, laſtly, let him [learn or] contemplate the Nature and Reſolution of Æquations.

## *Of the Signification of ſome Words and Notes.*

By *Number* we underſtand not ſo much a Multitude of Unities, as the abſtracted *Ratio* of any Quantity, to another Quantity of the ſame Kind, which we take for Unity.

[Number] is threefold; integer, fracted, and ſurd, to which laſt Unity is incommenſurable. Every one underſtands the Notes of whole Numbers, (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and the Values of thoſe Notes when more than one are ſet together. But as Numbers plac'd on the left Hand, next before Unity, denote Tens of Units, in the ſecond Place Hundreds, in the third Place Thouſands, &c. ſo Numbers ſet in the firſt Place after Unity, denote tenth Parts of an Unit, in the ſecond Place hundredth Parts, in the third thouſandth Parts, &c. and theſe are call'd *Decimal Fractions*, becauſe they always decreaſe in a *Decimal Ratio*; and to diſtinguiſh the Integers from the Decimals, we place a Comma, or a Point, or a ſeparating Line: Thus the Number 732⌊569 denotes ſeven hundred thirty two Units, together with five tenth Parts, ſix centeſimal, or hundredth Parts, and nine milleſimal, or thouſandth Parts of Unity. Which are alſo written thus 732,⌊569; or thus, 732.569; or alſo thus, 732⌊569, and ſo the Number 57104⌊2083 fifty ſeven thouſand one hundred and four Units, together with two tenth Parts, eight thouſandth Parts, and three ten thouſandth Parts of Unity; and the Number 0,064 denotes ſix centeſimals and four milleſimal Parts. The Notes of Surds and fracted Numbers are ſet down in the following [Pages].

When the Quantity of any Thing is unknown, or look'd upon as indeterminate, ſo that we can't expreſs it in Numbers, we denote it by ſome *Species*, or by ſome Letter. And if we conſider known Quantities as indeterminate, we denote them, for Diſtinction ſake, with the initial [or former] Letters of the Alphabet, as $a$, $b$, $c$, $d$, &c. and the unknown ones by the final ones, $z$, $y$, $x$, &c. Some ſubſtitute Conſonants or great Letters for known Quantities, and Vowels or little Letters for the unknown ones.

Quantities are either Affirmative, or greater than nothing; or Negative, or leſs than nothing. Thus in humane Affairs, Poſſeſſions or Stock may be call'd *affirmative* Goods, and Debts *negative ones*. And ſo in local Motion, Progreſſion may be call'd affirmative Motion, and Regreſſion negative Motion; becauſe the firſt augments, and the other diminiſhes [the Length of] the Way made. And after the ſame Manner in Geometry, if a Line drawn any certain Way be reckon'd for Affirmative, then a Line drawn the contrary Way may be taken for Negative: As if $AB$ be drawn to the right, and $BC$ to the left; and $AB$ be reckon'd Affirmative, then $BC$ will be Negative; becauſe in the drawing it diminiſhes $AB$, and reduces it either to a ſhorter, as $AC$, or to none, if $C$ chances to fall upon the Point $A$, or to a leſs than none, if $BC$ be longer than $AB$ from which it is taken [*vide Fig.* 1.] A negative Quantity is denoted by the Sign $-$; the Sign $+$ is prefix'd to an affirmative one; and $\mp$ denotes an uncertain Sign, and $\pm$ a contrary uncertain one.

In an Aggregate of Quantities the Note $+$ ſignifies, that the Quantity it is prefix'd to, is to be added, and the Note $-$, that it is to be ſubtracted. And we uſually expreſs theſe Notes by the Words *Plus* (or *more*) and *Minus* (or *leſs*). Thus $2+3$, or 2 more 3, denotes the Sum of the Numbers 2 and 3, that is 5. And $5-3$, or 5 leſs 3, denotes the Difference which ariſes by ſubducting 3 from 5, that is 2: And $-5+3$ ſignifies the Difference which ariſes from ſubducting 5 from 3, that is 2; and $6-1+3$ makes 8. Alſo $a+b$ denotes the Sum of the Quantities $a$ and $b$, and $a-b$ the Difference which ariſes by ſubducting $b$ from $a$; and $a-b+c$ ſignifies the Sum of that Difference, and of the Quantity $c$.

Suppoſe if $a$ be 5, $b$ 2, and $c$ 8, then $a+b$ will be 7, and $a-b$ 3. and $a-b+c$ will be 11. Alſo $2a+3a$ is $5a$, and $3b-2a-b+3a$ is $2b+a$; for $3b-b$ makes $2b$, and $-1a+3a$ makes $2a$, whoſe Aggregate, or Sum, is $1b+2a$, and ſo in others. Theſe Notes + and − are called *Signs*. And when neither is prefix'd, the Sign + is always to be underſtood.

*Multiplication*, properly ſo call'd, is that which is made by Integers, as ſeeking a new Quantity, ſo many times greater than the Multiplicand, as the Multiplyer is greater than Unity; but for want of a better Word *Multiplication* is alſo made Uſe of in Fractions and Surds, to find a new Quantity in the ſame *Ratio* (whatever it be) to the Multiplicand, as the Multiplier has to Unity. Nor is Multiplication made only by *abſtract* Numbers, but alſo by *concrete* Quantities, as by Lines, Surfaces, Local Motion, Weights, *&c.* as far as theſe may be conceiv'd to expreſs [or involve] the ſame Ratio's to ſome other known Quantity of the ſame Kind, eſteem'd as Unity, as Numbers do among themſelves. As if the Quantity $A$ be to be multiply'd by a Line of 12 Foot, ſuppoſing a Line of 2 Foot to be Unity, there will be produc'd by that Multiplication $6A$, or ſix times $A$, in the ſame manner as if $A$ were to be multiply'd by the abſtract Number 6; for $6A$ is in the ſame reaſon to $A$, as a Line of 12 Foot has to a Line of 2 Foot. And ſo if you were to multiply any two Lines, $AC$ and $AD$, by one another, take $AB$ for Unity, and draw $BC$, and parallel to it $DE$, and $AE$ will be the Product of this Multiplication; becauſe it is to $AD$ as $AC$, to Unity $AB$, [*vide Fig.* 2.] Moreover, Cuſtom has obtain'd, that the Geneſis or Deſcription of a Surface, by a Line moving at right Angles upon another Line, ſhould be called the Multiplication of thoſe two Lines. For tho' a Line, however multiply'd, cannot become a Surface, and conſequently this Generation of a Surface by Lines is very different from Multiplication, yet they agree in this, that the Number of Unities in either Line, multiply'd by the Number of Unities in the other, produces an abſtracted Number of Unities in the Surface comprehended under thoſe Lines, if the ſuperficial Unity be defin'd as it uſed to be, *viz.* a Square whoſe Sides are linear Unities. As if the right Line $AB$ conſiſt of four Unities, and $AC$ of three, then the Rectangle $AD$ will conſiſt of four times three, or 12 ſquare Unities, as from the Scheme will appear, [*vide Fig.* 3.] And there is the like Analogy of a Solid and a Product made by the continual Multiplication of three Quantities. And hence it is, that the Words to *multiply into*, the

*Content*, a *Rectangle*, a *Square*, a *Cube*, a *Dimenſion*, a *Side*, and the like, which are Geometrical Terms, are made Uſe of in Arithmetical Operations. For by a *Square*, or *Rectangle*, or a Quantity of two Dimenſions, we do not always underſtand a Surface, but moſt commonly a Quantity of ſome other Kind, which is produc'd by the Multiplication of two other Quantities, and very often a Line which is produc'd by the Multiplication of two other Lines. And ſo we call a *Cube*, or *Parallelopiped*, or a *Quantity of three Dimenſions*, that which is produc'd by two Multiplications. We ſay likewiſe the *Side* for a *Root*, and uſe *Ducere* in *Latin* inſtead of *Multiply*; and ſo in others.

A Number prefix'd before any Species, denotes that Species to be ſo often to be taken; thus $2a$ denotes two $a$'s, $3b$ three $b$'s, $15x$ fifteen $x$'s. Two or more Species, immediately connected together without any Signs, denote a Product or Quantity made by the Multiplication of all the Letters together. Thus $ab$ denotes a Quantity made by multiplying $a$ by $b$, and $abx$ denotes a Quantity made by multiplying $a$ by $b$, and the Product again by $x$. As ſuppoſe, if $a$ were 2, and $b$ 3, and $x$ 5, then $ab$ would be 6, and $abx$ 30. Among Quantities multiplying one another, take Notice, that the Sign ×, or the Word *by* or *into*, is made Uſe of to denote the Product ſometimes; thus $3\times5$, or 3 by or into 5 denotes 15; but the chief Uſe of theſe Notes is, when compound Quantities are multiply'd together; as if $\overline{y-2b}$ were to multiply $\overline{y+b}$; the Way is to draw a Line over each Quantity, and then write them thus, $\overline{y-2b}$ into $\overline{y+b}$, or $\overline{y-2b}\times\overline{y+b}$.

*Diviſion* is properly that which is made Uſe of for integer or whole Numbers, in finding a new Quantity ſo much leſs than the Dividend, as Unity is than the Diviſor. But becauſe of the Analogy, the Word may alſo be uſed when a new Quantity is ſought, that ſhall be in any ſuch *Ratio* to the Dividend, as Unity has to the Diviſor, whether that Diviſor be a Fraction or ſurd Number, or other Quantity of any other Kind. Thus to divide the Line $AE$ by the Line $AC$, $AB$ being Unity, you are to draw $ED$ parallel to $CB$, and $AD$ will be the Quotient, [*vide Fig.* 4.] Moreover, it is call'd *Diviſion*, by reaſon of the Similitude [it carries with it] when a Rectangle is divided by a given Line as a Baſe, in order thereby to know the Height.

One Quantity below another, with a Line interpos'd, denotes a Quotient, or a Quantity ariſing by the Diviſion of the

the upper Quantity by the lower. Thus $\frac{6}{2}$ denotes a Quantity ariſing by dividing 6 by 2, that is 3; and $\frac{5}{8}$ a Quantity ariſing by the Diviſion of 5 by 8, that is one eighth Part of the Number 5. And $\frac{a}{b}$ denotes a Quantity which ariſes by dividing $a$ by $b$; as ſuppoſe $a$ was 15 and $b$ 3, then $\frac{a}{b}$ would denote 5. Likewiſe thus $\frac{ab-bb}{a+x}$ denotes a Quantity ariſing by dividing $ab-bb$ by $a+x$; and ſo in others.

Theſe Sorts of Quantities are called *Fractions*, and the upper Part is call'd by the Name of the *Numerator*, and the lower is call'd the *Denominator*.

Sometimes the Diviſor is ſet before the divided Quantity, [or Dividend] and ſeparated from it by [a Mark reſembling] an Arch of a Circle. Thus to denote the Quantity which ariſes by the Diviſion of $\frac{axx}{a+b}$ by $a-b$, we write it thus, $a-b)\frac{axx}{a+b}$.

Altho' we commonly denote Multiplication by the immediate Conjunction of the Quantities, yet an Integer, [ſet] before a Fraction, denotes the Sum of both; thus $3\frac{1}{2}$ denotes three and a half.

If a Quantity be multiply'd by it ſelf, the Number of *Facts* or Products is, for Shortneſs ſake, ſet at the Top of the Letter. Thus for $aaa$ we write $a^3$, for $aaaa$ $a^4$, for $aaaaa$ $a^5$, and for $aaabb$ we write $a^3bb$, or $a^3b^2$: as, ſuppoſe if $a$ were 5 and $b$ be 2, then $a^3$ will be 5×5×5 or 125, and $a^4$ will be 5×5×5×5 or 625, and $a^3b^2$ will be 5×5×5×2×2 or 500. Where Note, that if a Number be written immediately between two Species, it always belongs to the former; thus the Number 3 in the Quantity $a^3bb$, does not denote that $bb$ is to be taken thrice, but that $a$ is to be thrice multiply'd by it ſelf. Note, moreover, that theſe Quantities are ſaid to be of ſo many Dimenſions, or of ſo high a Power or Dignity, as they conſiſt of Factors or Quantities multiplying one another; and the Number ſet [on forwards] at the top [of the Letter] is called the Index of thoſe Powers or Dimenſions; thus $aa$ is [a Quantity] of two Dimenſions, or of the 2d Power, and $a^3$ of three, as the Number 3 at the top denotes. $aa$ is alſo call'd a *Square*, $a^3$ a *Cube*, $a^4$ a [*Biquadrate*, or] *ſquared Square*, $a^5$ a *Quadrato-Cube*, $a^6$ a *Cubo-Cube*, $a^7$ a *Quadrato-Quadrato-Cube*, [or *Squared-Squared Cube*] and ſo on. N. B. *Sir* Iſaac *has not here taken any Notice of the more modern Way of expreſſing theſe Powers, by calling the Root, or* $a$, *the firſt* [*or ſimple*] *Power,* $a^2$ *the ſecond Power,* $a^3$ *the third Power,* &c. And the Quantity $a$, by whoſe Multiplication by it ſelf theſe Powers are generated, is called their *Root, viz.* it is the Square Root of the Square $aa$, the Cube Root of the Cube $aaa$, &c. But when a Root, multiply'd by it ſelf, produces a Square, and that Square, multiply'd again by the Root, produces a Cube, *&c.* it will be (by the Definition of Multiplication) as Unity to the Root; ſo that Root to the Square, and that Square to the Cube, *&c.* and conſequently the Square Root of any Quantity, will be a mean Proportional between Unity and that Quantity, and the Cube Root the firſt of two mean Proportionals, and the Biquadratick Root the firſt of three, and ſo on. Wherefore Roots have theſe two Properties or Affections, firſt, that by multiplying themſelves they produce the ſuperior Powers; 2dly, that they are mean Proportionals between thoſe Powers and Unity. Thus, 8 is the Square Root of the Number 64, and 4 the Cube Root of it, is hence evident, becauſe 8×8, and 4×4×4 make 64, or becauſe as 1 to 8, ſo is 8 to 64, and 1 is to 4 as 4 to 16, and as 16 to 64; and hence, if the Square Root of any Line, as *AB*, is to be extracted, produce it to *C*, and let *BC* be Unity; then upon *AC* deſcribe a Semicircle, and at *B* erect a Perpendicular, occurring to [or meeting] the Circle in *D*; then will *BD* be the Root, becauſe it is a mean Proportional between *AB* and Unity *BC*, [*vide Fig.* 5.]

To denote the Root of any Quantity, we uſe to prefix this Note $\surd$ for a Square Root, and this $\surd 3$ if it be a Cube Root, and this $\surd 4$ for a Biquadratick Root, *&c.* Thus $\surd 64$ denotes 8, and $\surd 3{:}64$ denotes 4; and $\surd aa$ denotes $a$; and $\surd ax$ denotes the Square Root of $ax$; and $\surd 3{:}4axx$ the Cube Root of $4axx$: As if $a$ be 3 and $x$ 12; then $\surd ax$ will be $\surd 36$, or 6; and $\surd 3{:}4axx$ will be $\surd 3{:}1728$, or 12. And when theſe Roots can't be [exactly] *found*, or extracted, the Quantities are call'd *Surds*, as $\surd ax$; or *Surd Numbers*, as $\surd 12$.

There are ſome, that to denote the Square or firſt Power, make Uſe of $q$, and of $c$ for the Cube, $qq$ for the Biquadrate, and $qc$ for the Quadrato-Cube, *&c.* After this Manner for the Square, Cube, and Biquadrate of $A$, they write $Aq$, $Ac$, $Aqq$, *&c.* and for the Cube Root of $abb - x^3$, they write $\surd c : abb - x^3$. Others make Uſe of other Sorts of Notes, but they are now almoſt out of Faſhion.

The Mark [or the Sign] $=$ ſignifies, that the Quantities on each Side of it are equal. Thus $x=b$ denotes $x$ to be equal to $b$.

The Note $::$ ſignifies that the Quantities on both Sides of it are Proportional. Thus $a.b::c.d$ ſignifies, that $a$ is to $b$ [in the ſame Proportion] as $c$ to $d$; and $a.b.e::c.d.f$ ſignifies that $a$, $b$, and $e$, are to one another reſpectively, as $c$, $d$, and $f$, are among themſelves; or that $a$ to $c$, $b$ to $d$, and $e$ to $f$, are in the ſame *Ratio*. Laſtly, the Interpretation of any Marks or Signs that may be compounded out of theſe, will eaſily be known by the Analogy [they bear to theſe]. Thus $\frac{3}{4}a^3bb$ denotes three quarters of $a^3bb$, and $3\frac{a}{c}$ ſignifies thrice $\frac{a}{c}$, and $7\sqrt{ax}$ ſeven times $\sqrt{ax}$. Alſo $\frac{a}{b}x$ denotes the Product of $x$ by $\frac{a}{b}$; and $\frac{5ee}{4a+9e}z^3$ denotes the Product made by multiplying $z^3$ by $\frac{5ee}{4a+9e}$, that is the Quotient ariſing by the Diviſion of $5ee$ by $4a+9e$; and $\frac{2a^3}{9c}\sqrt{ax}$, that which is made by multiplying $\sqrt{ax}$ by $\frac{2a^3}{9c}$, and $\frac{7\sqrt{ax}}{c}$ the Quotient ariſing by the Diviſion of $7\sqrt{ax}$ by $c$; and $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ the Quotient ariſing by the Diviſion of $8a\sqrt{cx}$ by the Sum of the Quantities $2a+\sqrt{cx}$. And thus $\frac{3axx-x^3}{a+x}$ denotes the Quotient ariſing by the Diviſion of the Difference $3axx-x^3$ by the Sum $a+x$, and $\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Root of that Quotient, & $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Product of the Multiplication of that Root by the Sum $2a+3c$. Thus alſo $\sqrt{\frac{1}{4}aa+bb}$ denotes the Root of the Sum of the Quantities $\frac{1}{4}aa$ and $bb$, and $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes the Root of the Sum of the Quantities $\frac{1}{2}a$ and $\sqrt{\frac{1}{4}aa+bb}$, and $\frac{2a^3}{aa-zz}\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes that Root multiply'd by $\frac{2a^3}{aa-zz}$, and ſo in other Caſes.

But note, that in Complex Quantities of this Nature, there is no Neceſſity of giving a particular Attention to, or bearing in your Mind the Signification of each Letter; it will ſuffice in general to underſtand, *e.g.* that $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ ſignifies the Root of the Aggregate [or Sum] of $\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}$, whatever that Aggregate may chance to be, when Numbers or Lines are ſubſtituted in the Room of Letters. And thus [it is as ſufficient to underſtand] that $\dfrac{\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}}{a-\sqrt{ab}}$ ſignifies the Quotient ariſing by the Diviſion of the Quantity $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ by the Quantity $a-\sqrt{ab}$, as much as if thoſe Quantities were ſimple and known, though at preſent one may be ignorant what they are, and not give any particular Attention to the Conſtitution or Signification of each of their Parts. Which I thought I ought here [to inſinuate or] admoniſh, leaſt young Beginners ſhould be frighted [or deterr'd] in the very Beginning, by the Complexneſs of the Terms.

## *Of* ADDITION.

THE Addition of Numbers, where they are not compounded, is [eaſy and] manifeſt of it ſelf. Thus it is at firſt Sight evident, that 7 and 9, or $7+9$, make 16, and $11+15$ make 26. But in [longer or] more compounded Numbers, the Buſineſs is perform'd by writing the Numbers in a Row downwards, or one under another, and ſingly collecting the Sums of the [reſpective] Columns. As if the Numbers 1357 and 172 are to be added, write either of them (ſuppoſe) 172 under the other 1357, ſo that the Units of the one, *viz.* 2, may exactly ſtand under the Units of the other, *viz.* 7, and the other Numbers of the one exactly under the correſpondent ones of the other, *viz.* the Place of Tens under Tens, *viz.* 7 under 5, and that of Hundreds, *viz.* 1, under the Place of Hundreds of the other, *viz.* 3. Then beginning at the right Hand, ſay, 2 and 7 make 9, which write underneath; alſo 7 and 5 make 12; the laſt of which two Numbers, *viz.* 2, write underneath, and reſerve

$$\begin{array}{r} 1357 \\ 172 \\ \hline 1529 \end{array}$$

in your Mind the other, *viz.* 1, to be added to the two next Numbers, *viz.* 1 and 3; then ſay 1 and 1 make 2, which being added to 3 they make 5, which write underneath, and there will remain only 1, the firſt Figure of the upper Row of Numbers, which alſo muſt be writ underneath; and then you have the whole Sum, *viz.* 1529.

Thus, to add the Numbers 87899+13403+885+1920 into one Sum, write them one under another, ſo that all the Units may make one Column, the Tens another, the Hundredths a third, and the Places of Thouſands a fourth, and ſo on. Then ſay, 5 and 3 make 8, and 8 + 9 make 17; then write 7 underneath, and the 1 add to the next Rank, ſaying 1 and 8 make 9, 9 + 2 make 11, and 11 + 9 makes 20; and having writ the 0 underneath, ſay again as before, 2 and 8 makes 10, and 10 + 9 make 19, and 19 + 4 make 23, and 23 + 8 make 31; then reſerving 3 [in your Memory] write down 1 as before, and ſay again, 3 + 1 make 4, 4 + 3 make 7, and 7 + 7 make 14, wherefore write underneath 4, and laſtly ſay, 1 + 2 make 3, and 3 + 8 make 11, which in the laſt Place write down, and you will have the Sum of them all.

87899
13403
1920
885
114107

After the ſame Manner we alſo add Decimals, as in the following Example may be ſeen:

630,953
51,0807
305,27
987,3037

Addition is perform'd in Algebraick Terms, [or Species] by connecting the Quantities to be added with their proper Signs; and moreover, by uniting into one Sum thoſe that can be ſo united. Thus $a$ and $b$ make $a+b$; $a$ and $-b$ make $a-b$; $-a$ and $-b$ make $-a-b$; $7a$ and $9a$ make $7a+9a$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $-a\sqrt{ac}+b\sqrt{ac}$, or $b\sqrt{ac}-a\sqrt{ac}$; for it is all one, in what Order ſoever they are written.

Affirmative Quantities which agree in [are of the ſame Sort of] Species, are united together, by adding the prefix'd Numbers that are multiply'd into thoſe Species. Thus $7a+9a$ make $16a$. And $11bc+15bc$ make $26bc$. Alſo 3
$c$

+

$+5\frac{a}{c}$ make $8\frac{a}{c}$; and $2\sqrt{ac}+7\sqrt{ac}$ make $9\sqrt{ac}$; and $6\sqrt{ab-xx}+7\sqrt{ab-xx}$ make $13\sqrt{ab-xx}$. And in like manner, $6\sqrt{3}+7\sqrt{3}$ make $13\sqrt{3}$. Moreover, $a\sqrt{ac}+b\sqrt{ac}$ make $\overline{a+b}\sqrt{ac}$, by adding together $a$ and $b$ as Numbers multiplying $\sqrt{ac}$. And ſo $\frac{\overline{2a+3c}\sqrt{3axx-x^3}}{a+x}+\frac{3a\sqrt{3axx-x^3}}{a+x}$ make $\frac{\overline{5a+3c}\sqrt{3axx-x^3}}{a+x}$ becauſe $2a+3c$ and $3a$ make $5a+3c$.

Affirmative Fractions, that have the ſame Denominator, are united [or added together] by adding their Numerators. Thus $\frac{1}{5}+\frac{2}{5}$ make $\frac{3}{5}$, and $\frac{2ax}{b}+\frac{3ax}{b}$ make $\frac{5ax}{b}$ and thus $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}+\frac{17a\sqrt{cx}}{2a+\sqrt{cx}}$ make $\frac{25a\sqrt{cx}}{2a+\sqrt{cx}}$, and $\frac{aa}{c}+\frac{bx}{c}$ make $\frac{aa+bx}{c}$.

Negative Quantities are added after the ſame Way as Affirmative. Thus $-2$ & $-3$ make $-5$; $-\frac{4ax}{b}$ & $-\frac{11ax}{b}$ make $-\frac{15ax}{b}$; $-a\sqrt{ax}$ and $-b\sqrt{ax}$ make $\overline{-a-b}\sqrt{ax}$. But when a Negative Quantity is to be added to an Affirmative one, the Affirmative muſt be diminiſh'd by a Negative one. Thus, 3 and $-2$ make 1; $\frac{11ax}{b}$ and $-\frac{4ax}{b}$ make $-\frac{7ax}{b}$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $\overline{b-a}\sqrt{ac}$. And note, that when the Negative Quantity is greater than the Affirmative, the Aggregate [or Sum] will be Negative. Thus 2 and $-3$ make $-1$; $-\frac{11ax}{b}$ and $\frac{4ax}{b}$ make $-\frac{7ax}{b}$ and $2\sqrt{ac}$ and $-7\sqrt{ac}$ make $-5\sqrt{ac}$.

In the Addition of a greater Number of Quantities, or more compounded ones, it will be convenient to obſerve the

the [Method or] Form of Operation we have laid down above in the Addition of Numbers. As if $17ax - 14a + 3$, and $4a + 2 - 8ax$, and $7a - 9ax$, were to be added together, diſpoſe them ſo in Columns, that the Terms that contain the ſame Species may ſtand in a Row one under another, *viz.* the Numbers 3 and 2 in one Column, the Species $-14a$, and $4a$, and $7a$, in another Column, and the Species $17ax$, and $-8ax$, and $-9ax$ in a third; then I add the Terms of each Column by themſelves, ſaying, 2 and 3 make 5, which I write underneath, then $7a$ and $4a$ make $11a$, and moreover $-14a$ make $-3a$, which I alſo write underneath; laſtly, $-9ax$, and $-8ax$ make $-17ax$, to which $17ax$ added makes 0. And ſo the Sum comes out $-3a + 5$. After the ſame Manner the Buſineſs is done in the following Examples:

$$\begin{array}{rrr} 17ax & -14a & +3 \\ -8ax & +4a & +2 \\ -9ax & +7a & \\ \hline * & -3a & +5 \end{array}$$

$$\begin{array}{rr} 12x & +7a \\ 7x & +9a \\ \hline 19x & +16a \end{array} \qquad \begin{array}{rr} 11bc & -7\sqrt{ac} \\ 15bc & +2\sqrt{ac} \\ \hline 26bc & -5\sqrt{ac} \end{array} \qquad \begin{array}{rrr} -\frac{4ax}{b} & +6\sqrt{3} & +\frac{1}{5} \\ +\frac{11ax}{b} & -7\sqrt{3} & +\frac{2}{5} \\ \hline \frac{7ax}{b} & -\sqrt{3} & +\frac{3}{5} \end{array}$$

$$\begin{array}{rrr} & -6xx & +\frac{3}{7}x \\ 5x^3 & & +\frac{5}{7}x \\ \hline 5x^3 & -6xx & +\frac{8}{7}x \end{array} \qquad \begin{array}{rrrrr} & & aay & +2a^3 & -\frac{a^4}{2y} \\ & -2ayy & -4aay & +a^3 & \\ y^3 & +2ayy & -\frac{1}{2}aay & & \\ \hline y^3 & * & -3\frac{1}{2}aay & +3a^3 & -\frac{a^4}{2y} \end{array}$$

$$\begin{array}{rrrrr} 5x^4 & +2ax^3 & & & \\ -3x^4 & -2ax^3 & & +8\frac{1}{4}a^3\sqrt{aa+xx} & \\ -2x^4 & & +5bx^3 & & -20a^3\sqrt{aa-xx} \\ & & -4bx^3 & -7\frac{1}{4}a^3\sqrt{aa+xx} & \\ \hline * & & bx^3 & +a^3\sqrt{aa+xx} & -20a^3\sqrt{aa-xx}. \end{array}$$

Of

## Of SUBTRACTION.

THE Invention of the Difference of Numbers [that are] not too much compounded, is of it ſelf evident; as if you take 9 from 17, there will remain 8. But in more compounded Numbers, Subtraction is perform'd by ſubſcribing [or ſetting underneath] the Subtrahend, and ſubtracting each of the lower Figures from each of the upper ones. Thus to ſubtract 63543 from 782579, having ſubſcrib'd 63543, ſay, 3 from 9 and there remains 6, which write underneath; then 4 from 7 and there remains 3, which write likewiſe underneath; then 5 from 5 and there remains nothing, which in like manner ſet underneath; then 3 comes to be taken from 2, but becauſe 3 is greater [than 2] you muſt borrow 1 from the next Figure 8, which ſet down, together with 2, makes 12, from which 3 may be taken, and there will remain 9, which write likewiſe underneath; and then when beſides 6 there is alſo 1 to be taken from 8, add the 1 to the 6, and the Sum 7 [being taken] from 8, there will be left 1, which in like manner write underneath. *Laſtly*, when in the lower [Rank] of Numbers there remains nothing to be taken from 7, write underneath the 7, and ſo you have the [whole] Difference 719036.

$$\begin{array}{r}782579\\ \underline{63543}\\ 719036\end{array}$$

But eſpecial Care is to be taken, that the Figures of the Subtrahend be [plac'd] or ſubſcrib'd in their [proper or] homogeneous Places; *viz.* the Units of the one under the Units of the other, and the Tens under the Tens, and likewiſe the Decimals under the Decimals, *&c.* as we have ſhewn in Addition. Thus, to take the Decimal 0,63 from the Integer 547, they are not to be diſpos'd thus $\begin{array}{r}547\\0{,}63\end{array}$, but thus $\begin{array}{lll}547 & & \\ & 0{,}63 & \end{array}$, *viz.* ſo that the 0's, which ſupplies the Place of Units in the Decimal, muſt be plac'd under the Units of the other Number. Then 0 being underſtood to ſtand in the empty Places of the upper Number, ſay, 3 from 0, which ſince it cannot be, 1 ought to be borrow'd from the foregoing Place, which will make 10, from which 3 is to be taken, and there remains 7, which write underneath. Then that 1 which was borrow'd added to 6 makes 7, and this is to be taken from 0 above it; but ſince that can't be, you muſt again borrow 1 from the foregoing Place to make 10; then 7 from 10 leaves 3, which in like manner is to be writ underneath; then that 1 being added to 0, makes 1, which 1 being taken from 7 leaves 6, which again write underneath. Then write the two Figures 54 (ſince nothing remains to be taken from them) underneath, and you'll have the Remainder 546,37.

$$\begin{array}{r} 547 \\ \underline{0{,}63} \\ 546{,}37 \end{array}$$

For Exerciſe ſake, we here ſet down ſome more Examples, both in Integers and Decimals:

$$\begin{array}{rrrrrr} 1673 & 1673 & 458074 & 35{,}72 & 46{,}5003 & 308{,}7 \\ \underline{1541} & \underline{1580} & \underline{9205} & \underline{14{,}32} & \underline{3{,}078} & \underline{25{,}74} \\ 132 & 93 & 448869 & 21{,}4 & 43{,}4223 & 282{,}96 \end{array}$$

If a greater Number is to be taken from a leſs, you muſt firſt ſubtract the leſs from the greater, and then prefix a negative Sign to it. As if from 1541 you are to ſubtract 1673, on the contrary I ſubtract 1541 from 1673, and to the Remainder 132 I prefix the Sign $-$.

In Algebraick Terms, Subtraction is perform'd by connecting the Quantities, after having chang'd all the Signs of the Subtrahend, and by uniting thoſe together which can be united, as we have done in Addition. Thus $+7a$ from $+9a$ leaves $9a-7a$ or $2a$; $-7a$ from $+9a$ leaves $+9a+7a$, or $16a$; $+7a$ from $-9a$ leaves $-9a-7a$, or $-16a$; and $-7a$ from $-9a$ leaves $-9a+7a$, or $-2a$; ſo $3\frac{a}{e}$ from $5\frac{a}{e}$ leaves $2\frac{a}{e}$; $7\sqrt{ac}$ from $2\sqrt{ac}$ leaves $-5\sqrt{ac}$; $\frac{2}{9}$ from $\frac{5}{9}$ leaves $\frac{3}{9}$; $-\frac{4}{7}$ from $\frac{3}{7}$ leaves $\frac{7}{7}$; $-\frac{2ax}{b}$ from $\frac{3ax}{b}$ leaves $\frac{5ax}{b}$; $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ from $\frac{-17a\sqrt{cx}}{2a+\sqrt{cx}}$ leaves $\frac{-25a\sqrt{cx}}{2a+\sqrt{cx}}$; $\frac{aa}{c}$ from $\frac{bx}{c}$ leaves $\frac{bx-aa}{c}$; $a-b$ from $2a+b$ leaves $2a+b-a+b$, or $a+2b$; $3az-zz+ac$ from $3az$ leaves $3az-3az+zz-ac$, or $zz - ac$; $\frac{2aa - ab}{c}$ from $\frac{aa + ab}{c}$ leaves $\frac{aa + ab - 2aa + ab}{c}$, or $\frac{-aa + 2ab}{c}$; and $\overline{a - x}\sqrt{ax}$ from $\overline{a + x}\sqrt{ax}$ leaves $\overline{a + x - a + x}\sqrt{ax}$, or $2x\sqrt{ax}$, and ſo in others. But where Quantities conſiſt of more Terms, the Operation may be manag'd as in Numbers, as in the following Examples:

$$\begin{array}{r} 12x + 7a \\ 7x + 9a \\ \hline 5x - 2a \end{array} \qquad \begin{array}{r} 15bc + 2\sqrt{ac} \\ -11bc + 7\sqrt{ac} \\ \hline 26bc - 5\sqrt{ac} \end{array} \qquad \begin{array}{l} 5x^3 + \frac{5}{7}x \\ 6x^2 - \frac{3}{7}x \\ \hline 5x^3 - 6xx + \frac{8}{7}x \end{array}$$

$$\begin{array}{l} \frac{11ax}{b} - 7\sqrt{3} + \frac{2}{5} \\ \frac{4ax}{b} - 6\sqrt{3} - \frac{1}{5} \\ \hline \frac{7ax}{b} - \sqrt{3} + \frac{3}{5} \end{array}$$

## *Of* MULTIPLICATION.

NUMBERS which ariſe [or are produc'd] by the Multiplication of any two Numbers, not greater than 9, are to be learnt [and retain'd] in the Memory: As that 5 into 7 makes 35, and that 8 by 9 makes 72, *&c.* and then the Multiplication of greater Numbers is to be perform'd after the ſame Rule in theſe Examples.

$$\begin{array}{r} 795 \\ 4 \\ \hline 3180 \end{array}$$

If 795 is to be multiply'd by 4, write 4 underneath, as you ſee here. Then ſay, 4 into 5 makes 20, whoſe laſt Figure, *viz.* 0, ſet under the 4, and reſerve the former 2 for the next Operation. Say moreover, 4 into 9 makes 36, to which add the former 2, and there is made 38, whoſe latter Figure 8 write underneath as before, and reſerve the former 3. Laſtly, ſay, 4 into 7 makes 28, to which add the former 3 and there is made 31, which being alſo ſet underneath, you'll have the Number 3180, which comes out by multiplying the whole 795 by 4.

Moreover, if 9043 be to be multiply'd by 2305, write either of them, *viz.* 2305 under the other 9043 as before, and multiply the upper 9043 firſt by 5, after the Manner ſhewn, and there will come out 45215; then by 0, and there will come out 0000; thirdly, by 3, and there will come out 27129; laſtly, by 2, and there will come out 18086. Then diſpoſe theſe Numbers ſo coming out in a deſcending Series, [or under one another] ſo that the laſt Figure of every lower Row ſhall ſtand one Place nearer to the left Hand than the laſt of the next ſuperior Row. Then add all theſe together, and there will ariſe 20844115, the Number that is made by multiplying the whole 9043 by the whole 2305.

$$\begin{array}{r} 9043 \\ 2305 \\ \hline 45215 \\ 0000\phantom{0} \\ 27129\phantom{00} \\ 18086\phantom{000} \\ \hline 20844115 \end{array}$$

In the ſame Manner Decimals are multiply'd, by Integers, or other Decimals, or both, as you may ſee in the following Examples:

$$\begin{array}{r} 72,4 \\ 29 \\ \hline 6516 \\ 1448\phantom{0} \\ \hline 2099,6 \end{array} \qquad \begin{array}{r} 50,18 \\ 2,75 \\ \hline 25090 \\ 35126\phantom{0} \\ 10036\phantom{00} \\ \hline 137,9950 \end{array} \qquad \begin{array}{r} 3,9025 \\ 0,0132 \\ \hline 78050 \\ 117075\phantom{0} \\ 39025\phantom{00} \\ \hline 0,05151300 \end{array}$$

But note, in the Number coming out [or the Product] ſo many Figures muſt be cut off to the right Hand for Decimals, as there are Decimal Figures both in the Multiplyer and the Multiplicand. And if by Chance there are not ſo many Figures in the Product, the deficient Places muſt be fill'd up to the left Hand with 0's, as here in the third Example.

Simple Algebraick Terms are multiply'd by multiplying the Numbers into the Numbers, and the Species into the Species, and by making the Product Affirmative, if both the Factors are Affirmative, or both Negative; and Negative if otherwiſe. Thus $2a$ into $3b$, or $-2a$ into $-3b$ make $6ab$, or $6ba$; for it is no Matter in what Order they are plac'd. Thus alſo $2a$ by $-3b$, or $-2a$ by $3b$ make $-6ab$. And thus, $2ac$ into $8bcc$ make $16abccc$, or $16abc^3$; and $7axx$ into $-12aaxx$ make $-84a^3x^4$; and $-16cy$ into $31ay^3$ make $-496acy^4$; and $-4z$

into $-3\sqrt{az}$ make $12z\sqrt{az}$. And ſo 3 into $-4$ make $-12$, and $-3$ into $-4$ make 12.

Fractions are multiply'd, by multiplying their Numerators by their Numerators, and their Denominators by their Denominators; thus $\frac{2}{5}$ into $\frac{3}{7}$ make $\frac{6}{35}$; and $\frac{a}{b}$ into $\frac{c}{d}$ make $\frac{ac}{bd}$; and $2\frac{a}{b}$ into $3\frac{c}{d}$ make $6+\frac{a}{b}+\frac{c}{d}$, or $6\frac{ac}{bd}$; and $\frac{3acy}{2bb}$ into $\frac{-7cyy}{4b^3}$ make $\frac{-21accy^3}{8b^5}$; and $\frac{-4z}{c}$ into $\frac{-3\sqrt{az}}{c}$ make $\frac{12z\sqrt{az}}{cc}$; and $\frac{a}{b}x$ into $\frac{c}{d}x^3$ make $\frac{ac}{bd}x^3$. Alſo 3 into $\frac{2}{5}$ make $\frac{6}{5}$, as may appear, if 3 be reduc'd to the Form of a Fraction, *viz.* $\frac{3}{1}$, by making Uſe of Unity for the Denominator. And thus $\frac{15aaz}{cc}$ into $2a$ make $\frac{30a^3z}{cc}$. Whence note by the Way, that $\frac{ab}{c}$ and $\frac{a}{c}b$ are the ſame; as alſo $\frac{abx}{c}$, $\frac{ab}{c}x$, and $\frac{a}{c}bx$, alſo $\frac{\overline{a+b}\sqrt{cx}}{a}$ and $\frac{a+b}{a}\sqrt{cx}$; and ſo in others.

Radical Quantities of the ſame Denomination (that is, if they are both Square Roots, or both Cube Roots, or both Biquadratick Roots, &c.) are multiply'd by multiplying the Terms together [and placing them] under the ſame Radical Sign. Thus $\sqrt{3}$ into $\sqrt{5}$ makes $\sqrt{15}$; and the $\sqrt{ab}$ into $\sqrt{cd}$ makes $\sqrt{abcd}$; and $\sqrt[3]{5ayy}$ into $\sqrt[3]{7ayz}$ makes $\sqrt[3]{35aay^3z}$; and $\sqrt{\frac{a^3}{c}}$ into $\sqrt{\frac{abb}{c}}$ makes $\sqrt{\frac{a^4bb}{cc}}$, that is † $\frac{aab}{c}$; and $2a\sqrt{az}$ into $3b\sqrt{az}$ makes $6ab\sqrt{aazz}$, that is $6aabz$; and $\frac{3xx}{\sqrt{ac}}$ into $\frac{-2x}{\sqrt{ac}}$ makes $\frac{-6x^3}{\sqrt{aacc}}$, that

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† *See the Chapter of* Notation.

that is $\frac{-6x^3}{ac}$; and $\frac{-4x\sqrt{ab}}{7a}$ into $\frac{-3dd\sqrt{5cx}}{10ee}$ makes $\frac{12ddx\sqrt{5abcx}}{70aee}$.

Quantities that conſiſt of ſeveral Parts, are multiply'd by multiplying all the Parts of the one into all the Parts of the other, as is ſhewn in the Multiplication of Numbers. Thus, $c-x$ into $a$ makes $ac-ax$, and $aa+2ac-bc$ into $a-b$ makes $a^3+2aac-aab-3bac+bbc$. For $aa+2ac-bc$ into $-b$ makes $-aab-2acb+bbc$, and into $a$ makes $a^3+2aac-abc$, the Sum whereof is $a^3+2aac-aab-3abc+bbc$. A Specimen of this Sort of Multiplication, together with other like Examples, you have underneath:

$$\begin{array}{r} aa+2ac-bc \\ a-b \\ \hline -aab-2abc+bbc \\ a^3+2aac-abc \qquad\qquad \\ \hline a^3+2aac-aab-3abc+bbc \end{array} \qquad \begin{array}{r} a+b \\ a+b \\ \hline ab+bb \\ aa+ab \qquad \\ \hline aa+2ab+bb \end{array}$$

$$\begin{array}{l} \qquad a+b \\ \qquad a-b \\ \hline \qquad -ab-bb \\ aa+ab \\ \hline aa \quad * \quad -bb \end{array} \qquad \begin{array}{r} yy+2ay-\frac{1}{2}aa \\ yy-2ay+aa \\ \hline aayy+2a^3y-\frac{1}{2}a^4 \\ -2ay^3-4aayy+a^3y \qquad \\ y^4+2ay^3-\frac{1}{2}aayy \qquad\qquad\qquad \\ \hline y^4 \quad * \quad -3\frac{1}{2}aayy+3a^3y-\frac{1}{2}a^4 \end{array}$$

$$\begin{array}{c} \frac{2ax}{c}-\sqrt{\frac{a^3}{c}} \\ 3a+\sqrt{\frac{abb}{c}} \\ \hline \frac{2ax}{c}\sqrt{\frac{abb}{c}}-\frac{aab}{c} \\ \frac{6aax}{c}-3a\sqrt{\frac{a^3}{c}} \\ \hline \frac{6aax}{c}-3a\sqrt{\frac{a^3}{c}}+\frac{2ax}{c}\sqrt{\frac{abb}{c}}-\frac{aab}{c}. \end{array}$$

## Of DIVISION.

DIVISION is perform'd in Numbers, by ſeeking how many times the Diviſor is contain'd in the Dividend, and as often ſubtracting, and writing ſo many Units in the Quotient; and by repeating that Operation upon Occaſion, as often as the Diviſor can be ſubtracted. Thus, to divide 63 by 7, ſeek how many times 7 is contain'd in 63, and there will come out preciſely 9 for the Quotient; and conſequently $\frac{63}{7}$ is equal to 9. Moreover, to divide 371 by 7, prefix the Diviſor 7, and beginning at the firſt Figures of the Dividend, coming as near them as poſſible, ſay, how many times 7 is contain'd in 37, and you'll find 5; then writing 5 in the Quotient, ſubtract 5 × 7, or 35, from 37, and there will remain 2, to which ſet the laſt Figure of the Dividend, *viz.* 1; and then 21 will be the remaining Part of the Dividend for the next Operation; ſay therefore as before, how many times 7 is contain'd in 21? and the Anſwer will be 3; wherefore writing 3 in the Quotient, take 3 × 7, or 21, from 21 and there will remain 0. Whence it is manifeſt, that 53 is preciſely the Number that ariſes from the Diviſion of 371 by 7.

7) 371 (53  
35  
21  
21  
0

And thus to divide 4798 by 23, firſt beginning with the initial Figures 47, ſay, how many times is 23 contain'd in 47? Anſwer 2; wherefore write 2 in the Quotient, and from 47 ſubtract 2 × 23, or 46, and there will remain 1, to which join the next Number of the Dividend, *viz.* 9, and you'll have 19 to work upon next. Say therefore, how many times is 23 contain'd in 19? Anſwer 0; wherefore write 0 in the Quotient; and from 19 ſubtract 0 × 23, or 0, and there remains 19, to which join the laſt Number 8, and you'll have 198 to work upon next. Wherefore

23) 4798 (208,6086, &c.  
46  
19  
00  
198  
184  
140  
138  
20  
00  
200  
184  
160

in the laſt Place ſay, how many times is 23 contain'd in 198 (which may be gueſs'd at from the firſt Figures of each, 2 and 19, by taking notice how many times 2 is contain'd in 19)? I anſwer 8; wherefore write 8 in the Quotient, and from 198 ſubtract 8 × 23, or 184, and there will remain 14 to be farther divided by 23; and ſo the Quotient will be 208$\frac{14}{23}$. And if this Fraction is not lik'd, you may continue the Diviſion in Decimal Fractions as far as you pleaſe, by adding always a Cypher to the remaining Number. Thus to the Remainder 14 add 0, and it becomes 140. Then ſay how many times 23 in 140? Anſwer 6; write therefore 6 in the Quotient; and from 140 ſubtract 6 × 23, or 138, and there will remain 2; to which ſet a Cypher (or 0) as before. And thus the Work being continu'd as far as you pleaſe, there will at length come out this Quotient, *viz.* 208,6086, &c.

After the ſame Manner the Decimal Fraction 3,5218 is divided, by the Decimal Fraction 46,1, and there comes out 0,07639, *&c.* Where note, that there muſt be ſo many Figures cut off in the Quotient, for Decimals, as there are more in the laſt Dividend than the Diviſor: As in this Example 5, becauſe there are 6 in the laſt Dividend, *viz.* 3,521800, and 1 in the Diviſor 46,1.

46,1) 3,5218 (0,07639
322,7
2948
2766
1820
1383
4370
4149
221

We have here ſubjoin'd more Examples, for Clearneſs ſake, *viz.*

9043) 20844115 (2305.
18086
27581
27129
45215
45215
0

72,4) 2099,6 (29
1448
6516
6516
0

50,18)

```
50,18) 137,995 (2,75          0,0132) 0,051513 (3,9025
       10036                          396
       -----                          ----
        37635                         1191
        35126                         1188
        -----                         ----
         25090                         339
         25090                         264
         -----                         ---
             0                          660
                                        660
                                        ---
                                          0
```

N. B. *The wording of this Rule in Sir* Iſaac, *ſeeming a little obſcure, this other equivalent Rule may be added,* viz. *Obſerve what is the Quality of that Figure in the Dividend under which the Place of Integer Units in the Diviſor does or ſhould ſtand; for the ſame will be the Quality of the firſt Figure of the Quotient,* e. g.

345) ,00468 (1

*In this Example,* 5 *being the Place of Integer Units in the Dividend, that ſet under the Dividend, ſo as to divide it, would fall under the Figure* 8, *which is the Place of Hundreds of Thouſandths in the Dividend; therefore the Unit in the Quotient muſt ſtand in the Place of Hundreds of Thouſandths; and to make it do ſo, four Cyphers muſt be plac'd before it,* viz. ,00001, &c. *is the true Quotient.*

In Algebraick Terms Diviſion is perform'd by the Reſolution of what is compounded by Multiplication. Thus, $ab$ divided by $a$ gives for the Quotient $b$. $6ab$ divided by $2a$ gives $3b$; and divided by $-2a$ gives $-3b$. $-6ab$ divided by $2a$ gives $-3b$, and divided by $-2a$ gives $3b$. $16abc^3$ divided by $2ac$ gives $8bcc$. $-84a^3x^4$ divided by $-12aaxx$ gives $7axx$. Likewiſe $\frac{6}{35}$ divided by $\frac{2}{5}$ gives $\frac{3}{7}$. $\frac{ac}{bd}$ divided by $\frac{a}{b}$ gives $\frac{c}{d}$. $\frac{-21accy^3}{8b^5}$ divided by $\frac{3acy}{2bb}$ gives $\frac{-7cyy}{4b^3}$. $\frac{6}{5}$ divided by 3 gives $\frac{2}{5}$; and reciprocally $\frac{6}{5}$ divided by $\frac{2}{5}$ gives $\frac{3}{1}$, or 3. $\frac{30a^3z}{cc}$

divided by $2a$ gives $\frac{15aaz}{cc}$; and reciprocally divided by $\frac{15aaz}{cc}$ gives $2a$. Likewiſe $\sqrt{15}$ divided by $\sqrt{3}$ gives $\sqrt{5}$. $\sqrt{abcd}$ divided by $\sqrt{cd}$ gives $\sqrt{ab}$. $\sqrt{a^{3}c}$ by $\sqrt{ac}$ gives $\sqrt{aa}$, or $a$. $\sqrt{}^{3}\,35aay^{3}z$ divided by $\sqrt{}^{3}\,5ayy$ gives $\sqrt{}^{3}\,7ayz$. $\frac{\sqrt{a^{4}bb}}{cc}$ divided by $\frac{\sqrt{a^{3}}}{c}$ gives $\frac{\sqrt{abb}}{c}$. $\frac{12ddx\sqrt{5abcx}}{70aee}$ divided by $\frac{-3dd\sqrt{5cx}}{10ee}$ gives $\frac{-4x\sqrt{ab}}{7a}$. And ſo $\overline{a+b}\sqrt{ax}$ divided by $a+b$ gives $\sqrt{ax}$; and reciprocally divided by $\sqrt{ax}$ gives $a+b$. And $\frac{a}{a+b}\sqrt{ax}$ divided by $\frac{1}{a+b}$ gives $a\sqrt{ax}$, or divided by $a$ gives $\frac{1}{a+b}\sqrt{ax}$, or $\frac{\sqrt{ax}}{a+b}$; and reciprocally divided by $\frac{\sqrt{ax}}{a+b}$ gives $a$. But in Diviſions of this Kind you are to take care, that the Quantities divided by one another be of the ſame Kind, *viz.* that Numbers be divided by Numbers, and Species by Species, Radical Quantities by Radical Quantities, Numerators of Fractions by Numerators, and Denominators by Denominators; alſo in Numerators, Denominators, and Radical Quantities, the Quantities of each Kind muſt be divided by homogeneous ones [or Quantities of the ſame Kind.]

Now if the Quantity to be divided cannot be divided by the Diviſor [propos'd], it is ſufficient to write the Diviſor underneath, with a Line between them. Thus to divide $ab$ by $c$, write $\frac{ab}{c}$; and to divide $\overline{a+b}\sqrt{cx}$ by $a$, write $\frac{\overline{a+b}\sqrt{cx}}{a}$, or $\frac{a+b}{a}\sqrt{cx}$. And ſo $\sqrt{ax-xx}$ divided by $\sqrt{cx}$ gives $\frac{\sqrt{ax-xx}}{\sqrt{cx}}$, or $\sqrt{\frac{ax-cx}{cx}}$. And $\overline{aa+ab}\sqrt{aa-2xx}$ divided by $\overline{a-b}\sqrt{aa-xx}$ gives $\frac{aa+ab}{a-b}\sqrt{\frac{ax-2xx.}{aa-xx}}$ And $12\sqrt{5}$ divided by $4\sqrt{7}$ gives $3\sqrt{\tfrac{5}{7}}$.

But when theſe Quantities are Fractions, multiply the Numerator of the Dividend into the Denominator of the Diviſor, and the Denominator into the Numerator, and the firſt Product will be the Numerator, and the latter the Denominator of the Quotient. Thus to divide $\frac{a}{b}$ by $\frac{c}{d}$ write $\frac{ad}{bc}$, that is, multiply $a$ by $d$ and $b$ by $c$. In like Manner, $\frac{3}{7}$ by $\frac{5}{4}$ gives $\frac{12}{35}$. And $\frac{3a}{4c}\sqrt{ax}$ divided by $\frac{2c}{5a}$ gives $\frac{15aa}{8cc}\sqrt{ax}$, and divided by $2c\frac{\sqrt{aa-xx}}{5a\sqrt{ax}}$ gives $\frac{15a^3x}{8cc\sqrt{aa-xx}}$. After the ſame Manner, $\frac{ad}{b}$ divided by $c$ (or by $\frac{c}{1}$) gives $\frac{ad}{bc}$. And $c$ (or $\frac{c}{1}$) divided by $\frac{ad}{b}$ gives $\frac{bc}{ad}$. And $\frac{3}{7}$ divided by $5$ gives $\frac{3}{35}$. And $3$ divided by $\frac{5}{4}$ gives $\frac{12}{5}$. And $\frac{a+b}{c}\sqrt{cx}$ divided by $a$ gives $\frac{a+b}{ac}\sqrt{cx}$. And $\overline{a+b}\sqrt{cx}$ divided by $\frac{a}{c}$ gives $\frac{ac+bc}{a}\sqrt{cx}$. And $2\sqrt{\frac{axx}{c}}$ divided by $3\sqrt{cd}$ gives $\frac{2}{3}\sqrt{\frac{axx}{ccd}}$; and divided by $3\sqrt{\frac{cd}{x}}$ gives $\frac{2}{3}\sqrt{\frac{ax^3}{ccd}}$. And $\frac{1}{5}\sqrt{\frac{7}{11}}$ divided by $\frac{1}{2}\sqrt{\frac{3}{7}}$ gives $\frac{2}{5}\sqrt{\frac{49}{33}}$, and ſo in others.

A Quantity compounded of ſeveral Terms, is divided by dividing each of its Terms by the Diviſor. Thus $aa + 3ax - xx$ divided by $a$ gives $a + 3x - \frac{xx}{a}$. But when the Diviſor conſiſts alſo of ſeveral Terms, the Diviſion is perform'd as in Numbers. Thus to divide $a^3 + 2aac - aab - 3abc + bbc$ by $a - b$, ſay, how many times is $a$ contain'd in $a^3$, *viz.* the firſt Term of the Diviſor in the firſt Term of the Dividend? Anſwer $aa$. Wherefore write $aa$ in the Quotient; and having ſubtracted $a - b$ multiply'd into $aa$, or $a^3 - aab$ from the Dividend, there will remain

$2aac-3abc+bbc$ yet to be divided. Then ſay again, how many times $a$ in $2aac$? Anſwer $2ac$. Wherefore write alſo $2ac$ in the Quotient, and having ſubtracted $a-b$ into $2ac$, or $2aac-2abc$ from the aforeſaid Remainder, there will yet remain $-abc+bbc$. Wherefore ſay again, how many times $a$ in $-abc$? Anſwer $-bc$, and then write $-bc$ in the Quotient; and having, in the laſt Place, ſubtracted $+a-b$ into $-bc$, *viz.* $-abc+bbc$ from the laſt Remainder, there will remain nothing; which ſhews that the Diviſion is at an End, and the Quotient coming out [juſt] $aa+2ac-bc$.

But that theſe Operations may be duly reduc'd to the Form which we uſe in the Diviſion of Numbers, the Terms both of the Dividend and the Diviſor muſt be diſpos'd in Order, according to the Dimenſions of that Letter which is [ofteneſt found or] judg'd moſt proper for the [Eaſe of the] Operation; ſo that thoſe Terms may ſtand firſt, in which that Letter is of moſt Dimenſions, and thoſe in the ſecond Place whoſe Dimenſions are next higheſt; and ſo on to thoſe wherein that Letter is not at all involv'd, [or into which it is not at all multiply'd] which ought to ſtand in the laſt Place. Thus, in the Example we juſt now brought, if the Terms are diſpos'd according to the Dimenſions of the Letter $a$, the following Diagram will ſhew the Form of the Work, *viz.*

$$\begin{array}{llllll}
a-b) & a^3 & \begin{matrix}+2aac\\-aab\end{matrix} & -3abc & +bbc & (aa+2ac-bc \\
\hline
 & a^3 & -aab & & & \\
\hline
 & 0 & +2aac & -3abc & & \\
 & & 2aac & -2abc & & \\
\hline
 & & 0 & -abc & +bbc & \\
 & & & -abc & +bbc & \\
\hline
 & & & 0 & 0 &
\end{array}$$

Where may be ſeen, that the Term $a^3$, or $a$ of three Dimenſions, ſtands in the firſt Place of the Dividend, and the Terms $\begin{matrix}2aac\\-aab\end{matrix}$, in which $a$ is of two Dimenſions, ſtand in the ſecond Place, and ſo on. The Dividend might alſo have been writ thus;

$a^3$

$$a^3 {+2c \atop -b} aa - 3bca + bbc.$$

Where the Terms that ſtand in the ſecond Place are united, by collecting together [or placing by one another] the Factors [or Coefficients] of the Letter [where it is] of the ſame Dimenſion. And thus, if the Terms were to be diſpos'd according to the Dimenſions of the Letter $b$, the Buſineſs muſt be perform'd [or would ſtand] as in the following Diagram, the Explication whereof we ſhall here ſubjoin:

$$\begin{array}{l}
-b+a)\; cbb {-3ac \atop -aa} b {+a^3 \atop +2aac} \left(-cb {+2ac \atop +aa}\right. \\
\qquad\quad \underline{cbb - acb} \\
\qquad\qquad 0 {-2ac \atop -aa} b {+a^3 \atop +2aac} \\
\qquad\qquad\;\; \underline{{-2ac \atop -aa} b {+2aac \atop +a^3}} \\
\qquad\qquad\qquad 0 \qquad 0
\end{array}$$

Say, How many times is $-b$ contain'd in $cbb$? Anſwer $-cb$. Wherefore having writ $-cb$ in the Quotient, ſubtract $\overline{-b+a} \times -cb$, or $bbc - abc$, and there will remain in the ſecond Place ${-2ac \atop -aa} b$. To this Remainder add, if you pleaſe, the Quantities [that ſtand] in the laſt Place, *viz.* ${a^3 \atop 2aac}$, and ſay again, how many times is $-b$ contain'd in ${-2ac \atop -aa} b$? Anſwer ${+2ac \atop +aa}$. Theſe therefore being writ in the Quotient, ſubtract $-b+a$ multiply'd by ${+2ac \atop +aa}$, or ${-2ac \atop -aa} b {+2aac \atop +a^3}$, and there will remain nothing. Whence it is manifeſt, that the Diviſion is at an End, the Quotient coming out $-cb + 2ac + aa$, as before.

And thus, if you were to divide $aay^4 - aac^4 + yyc^4 + y^6 - 2y^4cc - a^6 - 2a^4cc - a^4yy$ by $yy - aa - cc$. I order [or place] the Quantities according to the [Dimenſions of the] Letter $y$, thus:

$$yy\begin{matrix}-aa\\-cc\end{matrix}\Big)\,y^6\begin{matrix}+aa\\-2cc\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}yy\begin{matrix}-a^6\\-2a^4cc.\\-aac^4\end{matrix}$$

Then I divide as in the following Diagram.

Here are added other Examples, in which you are to take Notice, that where the Dimenſions of the Letter, which [this Method of] ordering ranges, don't always proceed in the ſame Arithmetical Progreſſion, but ſometimes [interruptedly, or] by Way of Skipping, in the defective Places we note [this Mark] ✱.

$$yy\begin{matrix}-aa\\-cc\end{matrix}\Big)\,y^6\begin{matrix}+aa\\-2cc\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}yy\begin{matrix}-a^6\\-2a^4cc.\\-aac^4\end{matrix}$$

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$y^6\begin{matrix}-aa\\-cc\end{matrix}y^4\qquad\Big(\,y^4\begin{matrix}+2aa\\-cc\end{matrix}y^2\begin{matrix}+a^4\\+aacc.\end{matrix}$$

$$0\begin{matrix}+2aa\\-cc\end{matrix}y^4\begin{matrix}-2a^4\\-aacc\\+c^4\end{matrix}y^2$$

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$0\qquad\begin{matrix}+a^4\\+aacc\end{matrix}y^2$$

$$\begin{matrix}+a^4\\+aacc\end{matrix}y^2\begin{matrix}-a^6\\-2a^4cc\\-aac^4\end{matrix}$$

$$\underline{\qquad\qquad\qquad\qquad}$$

$$0\qquad\qquad 0$$

$$a+b)\,aa\ast-bb\,(a-b$$

$$\underline{aa+ab}$$

$$0-ab$$

$$\underline{-ab-bb}$$

$$0\qquad 0$$

$yy = 2ay$

$$\begin{array}{llllll}
yy-2ay+aa\,) & y^4\ * & -3\tfrac{1}{2}aayy & +3a^3y & -\tfrac{1}{2}a^4 & (\,yy+2ay-\tfrac{1}{2}aa \\
 & y^4 & -2ay^3 & +aayy & & \\
\hline
 & 0 & +2ay^3 & -4\tfrac{1}{2}aayy & & \\
 & & +2ay^3 & -4\ aayy & +2a^3y & \\
\hline
 & & 0 & -\tfrac{1}{2}aayy & +a^3y & \\
 & & & -\tfrac{1}{2}aayy & +a^3y & -\tfrac{1}{2}a^4 \\
\hline
 & & & 0 & 0 & 0
\end{array}$$

$$\begin{array}{lllllll}
aa+ab\sqrt{2}+bb\,) & a^4 & * & * & * & +b^4 & (\,aa-ab\sqrt{2}+bb \\
 & a^4 & +a^3b\sqrt{2} & +aabb & & & \\
\hline
 & & -a^3b\sqrt{2} & -aabb & & & \\
 & & -a^3b\sqrt{2} & -2aabb & -ab^3\sqrt{2} & & \\
\hline
 & & & +aabb & +ab^3\sqrt{2} & & \\
 & & & +aabb & +ab^3\sqrt{2} & +b^4 & \\
\hline
 & & & 0 & 0 & 0 &
\end{array}$$

Some begin Diviſion from the laſt Terms, but it comes to the ſame Thing, if, inverting the Order of the Terms, you begin from the firſt. There are alſo other Methods of dividing, but it is ſufficient to know the moſt eaſy and commodious.

## *Of* EXTRACTION *of* ROOTS.

WHEN the Square Root of any Number is to be extracted, it is firſt to be noted with Points in every other Place, beginning from Unity; then you are to write down ſuch a Figure for the Quotient, or Root, whoſe Square ſhall be equal to, or neareſt, leſs than the Figure, or Figures to the firſt Point. And [then] ſubtracting that Square, the other Figures of the Root will be found one by one, by dividing the Remainder by the double of the Root as far as extracted, and each Time taking from that Remainder the

Square of the Figure that laſt came out, and the Decuple of the aforeſaid Diviſor augmented by that Figure.

Thus to extract the Root out of 99856, firſt Point it after this Manner, 99856, then ſeek a Number whoſe Square ſhall equal the firſt Figure 9, *viz.* 3, and write it in the Quotient; and then having ſubtracted from 9, 3×3, or 9, there will remain 0; to which ſet down the Figures to the next Point, *viz.* 98 for the following Operation. Then taking no Notice of the laſt Figure 8, ſay, How many times is the Double of 3, or 6, contain'd in the firſt Figure 9? Anſwer 1; wherefore having writ 1 in the Quotient, ſubtract the Product of 1 × 61, or 61, from 98, and there will remain 37, to which connect the laſt Figures 56, and you'll have the Number 3756, in which the Work is next to be carry'd on. Wherefore alſo neglecting the laſt Figure of this, *viz.* 6, ſay, How many times is the double of 31, or 62, contain'd in 375, (which is to be gueſs'd at from the initial Figures 6 and 37, by taking Notice how many times 6 is contain'd in 37?) Anſwer 6; and writing 6 in the Quotient, ſubtract 6×626, or 3756, and there will remain 0; whence it appears that the Buſineſs is done; the Root coming out 316.

```
99856 (316
9
-----
 098
  61
 ----
 3756
 3756
 ----
    0
```

*Otherwiſe with the Diviſors ſet down it will ſtand thus:*

```
    99856 (316
    9
    -----
  6)98
    61
  -------
62)3756
    3756
   -----
      0
```

*And ſo in others.*

And ſo if you were to extract the Root out of 22178791, firſt having pointed it, ſeek a Number whoſe Square (if it cannot be [exactly] equall'd) ſhall be the next leſs Square (or neareſt) to 22, the Figures to the firſt Point, and you'll find it to be 4: For 5×5, or 25, is greater than 22; and 4×4, or 16, leſs; wherefore 4 will be the firſt Figure of the Root. This therefore being writ in the Quotient, from 22 take the Square 4×4, or 16, and to the Remainder 6 adjoin moreover the next Figures 17, and you'll have 617, from whoſe Diviſion by the double of 4 you are to obtain the ſecond Figure of the Root, *viz.* neglecting the laſt Figure 7, ſay, how many times is 8 contain'd in 61? Anſwer 7; wherefore write 7 in the Quotient, and from 617 take the Product of 7 into 87, or 609, and there will remain 8, to which join the two next Figures 87, and you'll have 887, by the Diviſion whereof by the double of 47, or 94, you are to obtain the third Figure; as ſay, How many times is 94 contain'd in 88? Anſwer 0; wherefore write 0 in the Quotient, and adjoin the two laſt Figures 91, and you'll have 88791, by whoſe Diviſion by the double of 470, or 940, you are to obtain the laſt Figure, *viz.* ſay, How many times 940 in 8879? Anſwer 9; wherefore write 9 in the Quotient, and you'll have the Root 4709.

```
22178791 (4709,43637, &c.
16
----
 617
 609
------
   88791
   84681
   ------
   411000
   376736
   -------
    3426400
    2825649
    --------
     60075100
     56513196
     ---------
     356190400
     282566169
     ---------
      73624231
```

But ſince the Product 9×9409, or 84681, ſubtracted from 88791, leaves 4110, that is a Sign that the Number 4709 is not the Root of the Number 22178791 preciſely, but that it is a little leſs. And in this Caſe, and in others like it, if you deſire the Root ſhould approach nearer, you muſt [proceed or] carry on the Operation in Decimals, by adding to the Remainder two Cyphers in each Operation. Thus the Remainder 4110 having two Cyphers added to it, becomes 411000; by the Diviſion whereof by the double of 4709, or 9418, you'll have the firſt Decimal Figure 4. Then having writ 4 in the Quotient, ſubtract 4×94184, or 376736 from 411000, and there will remain 34264. And ſo having added two more Cyphers, the Work may be carry'd on at Pleaſure, the Root at length coming out 4709,43637, &c.

But when the Root is carry'd on half-way, or above, the reſt of the Figures may be obtain'd by Diviſion alone. As in this Example, if you had a Mind to extract the Root to nine Figures, after the five former 4709,4 are extracted, the four latter may be had, by dividing the Remainder by the double of 4709,4.

And after this Manner, if the Root of 32976 was to be extracted to five Places in Numbers: After the Figures are pointed, write 1 in the Quotient, as [being the Figure] whoſe Square 1×1, or 1, is the greateſt that is contain'd in 3 the Figure to the firſt Point; and having taken the Square of 1 from 3, there will remain 2; then having ſet the two next Figures, *viz.* 29 to it, (*viz.* to 2) ſeek how many times the double of 1, or 2, is contain'd in 22, and you'll find indeed that it is contain'd more than 10 times; but you are never to take your Diviſor 10 times, no, nor 9 times in this Caſe; becauſe the Product of 9×29, or 261, is greater than 229, from which it would be to be taken [or ſubtracted]. Wherefore write only 8: And then having writ 8 in the Quotient, and ſubtracted 8×28, or 224, there will remain 5; and having ſet down to this the Figures 76, ſeek how many times the double of 18, or 36, is contain'd in 57, and you'll find 1, and ſo write 1 in the Quotient; and having ſubtracted 1×361, or 361 from 576, there will remain 215. Laſtly, to obtain the remaining Figures, divide this Number 215 by the double of 181, or 362, and you'll have the Figures 59, which being writ in the Quotient, you'll have the Root 181,59.

```
     32976 (181,59
     1
  2)229
    224
 36) 576
     361
362)215 (59, &c.
```

After the ſame Way Roots are alſo extracted out of Decimal Numbers. Thus the Root of 329,76 is 18,159; and the Root of 3,2976 is 1,8159; and the Root of 0,032976 is 0,18159, and ſo on. But the Root of 3297,6 is 57,4247; and the Root of 32,976 is 5,74247. And thus the Root of 9,9856 is 3,16. But the Root of 0,99856 is 0,999279, &c. as will appear from the following Diagrams:

3297,60

```
         3297,60 (57,4247                         9,9856 (3,16
         25                                       9
10)      797                                   6) 98
         749                                      61
114)     4860                                 62) 3756
           4576                                   3756
1148)    28400                                    0
           22964
11484)   543600
           459376
114848)  8422400
           8039409
            382991

                      0,998560 (0,999279
                       81
                 18)   1885
                       1701
                198)   18460
                        17901
               1998)     55900
                         39964
              19984)     1593600
                         1398929
             199854)      19467100
                          17986941
                           1480159
```

I will comprehend the Extraction of the Cubick Root, and of all others, under one general Rule, conſulting rather the Eaſe of the Praxis than the Expeditiouſneſs of it, leſt I ſhould [too much] retard [the Learner] in Things that are of no frequent Uſe, *viz.* every third Figure beginning from Unity is firſt of all to be pointed, if the Root [to be extracted] be a Cubick one; or every fifth, if it be a Quadrato-Cubick [or of the fifth Power], and then ſuch a Figure is to be writ in the Quotient, whoſe greateſt Power (*i. e.* whoſe Cube, if it be a Cubick Power, or whoſe Quadrato-Cube, if it be the fifth Power, *&c.*) ſhall either be equal to the Figure or Figures before the firſt Point, or next leſs [under them]; and then having ſubtracted that Power, the next Figure will be found by dividing the Remainder augmented by the next Figure of the Reſolvend, by the next leaſt Power of the Quotient, multiply'd by the Index of the Power to be extracted, that is, by the triple Square, if the Root be a Cubick one; or by the quintuple Biquadrate [*i. e.* five times the Biquadrate] if the Root be of the fifth Power, *&c.* And having again ſubtracted the Power of the whole Quotient from the firſt Reſolvend, the third Figure will be found by dividing that Remainder augmented by the next Figure of the Reſolvend, by the next leaſt Power of the whole Quotient, multiply'd by the Index of the Power to be extracted.

Thus to extract the Cube Root of 13312053, the Number is firſt to be pointed after this Manner, *viz.* $1\dot{3}31\dot{2}05\dot{3}$. Then you are to write the Figure 2, whoſe Cube is 8, in the [firſt Place of] the Quotient, as which is the next leaſt [Cube] to the Figures 13, [which is not a perfect Cube Number] or to the firſt Point; and having ſubtracted that Cube, there will remain 5; which being augmented by the next Figure of the Reſolvend 3, and divided by the triple Square of the Quotient 2, by ſeeking how many times 3 × 4, or 12, is contain'd in 53, it gives 4 for the ſecond Figure of the Quotient. But ſince the Cube of the Quotient 24, *viz.* 13824 would come out too great to be ſubtracted from the Figures 13312 that preceed the ſecond Point, there muſt only 3 be writ in the Quotient: Then the Quotient 23 being in a ſeparate Paper, [or Place] multiply'd by 23 gives the Square 529, which again multiply'd by 23 gives the Cube 12167, and this taken from 13312, will leave 1145; which augmented by the next Figure of the Reſolvend 0, and divided by the triple Square of the Quotient 23, *viz.* by ſeeking how many times 3 × 529, or 1587, is contain'd in 11450, it gives 7 for the third Figure of the Quotient. Then the Quotient 237, multiply'd by 237, gives the

$1\dot{3}31\dot{2}05\dot{3}$ (237
Subtract the Cube 8

12) rem. 53 (4 or 3

Subtract Cube 12167
1587) rem. 11450 (7

Subtract 13312053
Remains 0

the Square 56169, which again multiply'd by 237 gives the Cube 13312053, and this taken from the Reſolvend leaves 0. Whence it is evident that the Root ſought is 237.

And ſo to extract the Quadrato-Cubical Root of 36430820, it muſt be pointed over every fifth Figure, and the Figure 3, whoſe Quadrato-Cube [or fifth Power] 243 is the next leaſt to 364, *viz.* to the firſt Point, muſt be writ in the Quotient. Then the Quadrato-Cube 243 being ſubtracted from 364, there remains 121, which augmented by the next Figure of the Reſolvend, *viz.* 3, and divided by five times the Biquadrate of the Quotient, *viz.* by ſeeking how many times 5×81, or 405, is contain'd in 1213, it gives 2 for the ſecond Figure. That Quotient 32 being thrice multiply'd by it ſelf, makes the Biquadrate 1048576; and this again multiply'd by 32, makes the Quadrato-Cube 33554432, which being ſubtracted from the Reſolvend leaves 2876388. Therefore 32 is the Integer Part of the Root, but not the true Root; wherefore, if you have a Mind to proſecute the Work in Decimals, the Remainder, augmented by a Cypher, muſt be divided by five times the aforeſaid Biquadrate of the Quotient, by ſeeking how many times 5×1048576, or 5242880, is contain'd in 2876388,0, and there will come out the third Figure, or the firſt Decimal 5. And ſo by ſubtracting the Quadrato-Cube of the Quotient 32,5 from the Reſolvend, and dividing the Remainder by five times its Biquadrate, the fourth Figure may be obtain'd. And ſo on *in Infinitum*.

$$\begin{array}{rl} & \underline{36430820}\ (32,5 \\ & 243 \\ 405) & \underline{1213}\ (2 \\ & 33554432 \\ 5242880) & 2876388,0\ (5 \end{array}$$

When the Biquadratick Root is to be extracted, you may extract twice the Square Root, becauſe $\sqrt{}^{4}$ is as much as $\sqrt{}^{2}\sqrt{}^{2}$. And when the Cubo-Cubick Root is to be extracted, you may firſt extract the Cube-Root, and then the Square-Root of that Cube-Root, becauſe the $\sqrt{}^{6}$ is the ſame as $\sqrt{}^{3}:\sqrt{}^{3}$; whence ſome have call'd theſe Roots not Cubo-Cubick ones, but Quadrato-Cubes. And the ſame is to be obſerv'd in other Roots, whoſe Indexes are not prime Numbers.

The Extraction of Roots out of ſimple Algebraick Quantities, is evident, even from [the Nature or Marks of] Notation it ſelf; as that $\sqrt{aa}$ is $a$, and that $\sqrt{aacc}$ is $ac$, and that $\sqrt{9aacc}$ is $3ac$, and that $\sqrt{49a^4xx}$ is $7aax$. And alſo that $\sqrt{\frac{a^4}{cc}}$, or $\frac{\sqrt{a^4}}{\sqrt{cc}}$ is $\frac{aa}{c}$, and that $\sqrt{\frac{a^4bb}{cc}}$ is $\frac{aab}{c}$, and that $\sqrt{\frac{9aazz}{25bb}}$ is $\frac{3az}{5b}$, and that $\sqrt{\frac{4}{9}}$ is $\frac{2}{3}$, and that $\sqrt[3]{\frac{8b^6}{27a^3}}$ is $\frac{2bb}{3a}$, and that $\sqrt[4]{aabb}$ is $\sqrt{ab}$. Moreover, that $b\sqrt{aacc}$, or $b$ into $\sqrt{aacc}$, is $b$ into $ac$ or $abc$. And that $3c\sqrt{\frac{9aazz}{25bb}}$ is $3c\times\frac{3az}{5b}$, or $\frac{9acz}{5b}$. And that $\frac{a+3x}{c}\sqrt{\frac{4bbx^4}{81aa}}$ is $\frac{a+3x}{c}\times\frac{2bxx}{9a}$, or $\frac{2abxx+6bx^3}{9ac}$.

I ſay, theſe are all evident, becauſe it will appear, at firſt Sight, that the propos'd Quantities are produc'd by multiplying the Roots into themſelves (as $aa$ from $a\times a$, $aacc$ from $ac$ into $ac$, $9aacc$ from $3ac$ into $3ac$, &c.) But when Quantities conſiſt of ſeveral Terms, the Buſineſs is perform'd as in Numbers. Thus, to extract the Square Root out of $aa+2ab+bb$, in the firſt Place, write the Root of the firſt Term $aa$, viz. $a$ in the Quotient, and having ſubtracted its Square $a\times a$, there will remain $2ab+bb$ to find the Remainder of the Root by. Say therefore, How many times is the double of the Quotient, or $2a$, contain'd in the firſt Term of the Remainder $2ab$? I anſwer $b$ [times], therefore write $b$ in the Quotient, and having ſubtracted the Product of $b$ into $\overline{2a+b}$, or $2ab+bb$, there will remain nothing. Which ſhews that the Work is finiſh'd, the Root coming out $a+b$.

$$\begin{array}{rl}
 & aa+2ab+bb\ (a+b \\
 & \underline{aa\qquad\qquad} \\
\text{c.} & +2ab+bb \\
 & \underline{+2ab+bb} \\
 & \quad 0 \qquad 0
\end{array}$$

And thus, to extract the Root out of $a^4+6a^3b+5a^2bb-12ab^3+4b^4$, firſt, ſet in the Quotient the Root of the firſt Term $a^4$, viz. $aa$, and having ſubtracted its Square $aa\times aa$, or $a^4$, there will remain $6a^3b+5aabb-12ab^3+4b^4$ to find the Remainder of the Root. Say therefore, How many times is $2aa$ contain'd in $6a^3b$? Anſwer $3ab$; wherefore write $3ab$ in the Quotient, and having ſubtracted the Product of $3ab$ into $2aa+3ab$, or $6a^3b+9aabb$, there will yet remain $-4aabb-12ab^3+4b^4$ to carry on the Work. Therefore ſay again, How many times is the double of the Quotient, *viz.* $2aa+6ab$ contain'd in $-4aabb-12ab^3$, or, which is the ſame Thing, ſay, How many times is the double of the firſt Term of the Quotient, or $2aa$, contain'd in the firſt Term of the Remainder $-4aabb$? Anſwer $-2bb$. Then having writ $-2bb$ in the Quotient, and ſubtracted the Product $-2bb$ into $2aa+6ab-2bb$, or $-4aabb-12ab^3+4b^4$, there will remain nothing. Whence it follows, that the Root is $aa+3ab-2bb$.

$$\begin{array}{llllll}
a^4 & +6a^3b & +5aabb & -12ab^3 & +4b^4 & (aa+3ab-2bb \\
a^4 & & & & & \\
\hline
 & 6a^3b & +5aabb & -12ab^3 & +4b^4 & \\
0. & +6a^3b & +9aabb & & & \\
\cline{2-3}
 & 0 & -4aabb & -12ab^3 & +4b^4 & \\
 & & -4aabb & -12ab^3 & +4b^4 & \\
\cline{3-5}
 & & 0 & 0 & 0 &
\end{array}$$

And thus the Root of the Quantity $xx-ax+\frac{1}{4}aa$ is $x-\frac{1}{2}a$; and the Root of the Quantity $y^4+4y^3-8y+4$ is $yy+2y-2$; and the Root of the Quantity $16a^4-24aaxx+9x^4+12bbxx-16aabb+4b^4$ is $3xx-4aa+2bb$, as may appear by the Diagrams underneath:

$$\begin{array}{llll}
xx & -ax & +\frac{1}{4}aa & (x-\frac{1}{2}a \\
xx & & & \\
\hline
0 & -ax & +\frac{1}{4}aa & \\
 & -ax & +\frac{1}{4}aa & \\
\hline
 & 0 & 0 &
\end{array}$$

$$\begin{array}{llll}
9x^4 & \begin{matrix}-24aa\\+12bb\end{matrix}\,x^2 & \begin{matrix}+16a^4\\-16aab^2\\+4b^4\end{matrix} & \left(3x^2\begin{matrix}-4aa\\+2bb\end{matrix}\right. \\
9x^4 & & & \\
\hline
0 & \begin{matrix}-24aa\\+12bb\end{matrix}\,x^2 & \begin{matrix}+16a^4\\-16a^2b^2\\+4b^4\end{matrix} & \\
\hline
 & 0 & 0 &
\end{array}$$

$$\begin{array}{llll}
y^4 + 4y^3 * - 8y + 4\ (yy + 2y - 2 & & & \\
y^4 & & & \\
\hline
0. & & & \\
\quad 4y^3 & + 4yy & & \\
\hline
\quad 0 & - 4yy & & \\
 & - 4yy & - 8y & + 4 \\
\hline
 & 0 & 0 & 0
\end{array}$$

If you would extract the Cube Root of $a^3 + 3aab + 3abb + b^3$, the Operation is [perform'd] thus:

$$\begin{array}{lllll}
 & a^3 + 3aab + 3abb + b^3\ (a + b & & & \\
 & a^3 & & & \\
\cline{2-2}
3aa) & 0 & + 3aab\ (b & & \\
\cline{2-4}
 & a^3 & + 3aab & + 3abb & + b^3 \\
 & 0 & 0 & 0 & 0
\end{array}$$

Extract firſt the Cube Root of the firſt Term $a^3$, viz. $a$, and ſet it down in the Quotient: Then, ſubtracting its Cube $a^3$, ſay, How many times is its triple Square, or $3aa$, contain'd in the next Term of the Remainder $3aab$? and there comes out $b$; wherefore write $b$ in the Quotient, and ſubtracting the Cube of the Quotient, there will remain 0. Therefore $a + b$ is the Root.

After the ſame Manner, if the Cube Root is to be extracted out of $z^6 + 6z^5 - 40z^3 + 96z - 64$, it will come out $zz + 2z - 4$. And ſo in higher Roots.

## *Of the* REDUCTION *of* FRACTIONS *and* RADICAL [*Quantities.*]

THE Reduction of Fractions and Radical Quantities is of Uſe in the preceding Operations, and is [of reducing them] either to the leaſt Terms, or to the ſame Denomination.

Of

# *Of the* REDUCTION *of* FRACTIONS *to the least Terms.*

FRACTIONS are reduc'd to the leaſt Terms by dividing the Numerators and Denominators by the greateſt common Diviſor. Thus the Fraction $\frac{aac}{bc}$ is reduc'd to a more Simple one $\frac{aa}{b}$ by dividing both $aac$ and $bc$ by $c$; and $\frac{203}{667}$ is reduc'd to a more Simple one $\frac{7}{23}$ by dividing both 203 and 667 by 29; and $\frac{203aac}{667bc}$ is reduc'd to $\frac{7aa}{23b}$ by dividing by $29c$. And ſo $\frac{6a^3-9acc}{6aa+3ac}$ becomes $\frac{2aa-3cc}{2a+c}$ by dividing by $3a$. And $\frac{a^3-aab+abb-b^3}{aa-ab}$ becomes $\frac{aa+bb}{a}$ by dividing by $a-b$.

And after this Method, the Terms after Multiplication or Diviſion may be for the moſt part abridg'd. As if you were to multiply $\frac{2ab^3}{3ccd}$ by $\frac{9acc}{bdd}$, or divide it by $\frac{bdd}{9acc}$, there will come out $\frac{18aab^3cc}{3bccd^3}$, and by Reduction $\frac{6aabb}{d^3}$. But in theſe Caſes, it is better to abbreviate the Terms before the Operation, by dividing thoſe Terms [firſt] by the greateſt common Diviſor, which you would be oblig'd to do afterwards. Thus, in the Example before us, if I divide $2ab^3$ and $bdd$ by the common Diviſor $b$, and $3ccd$ and $9acc$ by the common Diviſor $3cc$, there will come out the Fraction $\frac{2abb}{d}$ to be multiply'd by $\frac{3a}{dd}$, or to be divided by $\frac{dd}{3a}$, there coming out $\frac{6aabb}{d^3}$ as above. And ſo $\frac{aa}{c}$ into $\frac{c}{b}$ becomes $\frac{aa}{1}$ into $\frac{1}{b}$, or $\frac{aa}{b}$. And $\frac{aa}{c}$ divided by $\frac{b}{c}$

becomes $aa$ divided by $b$, or $\frac{aa}{b}$. And $\frac{a^3-axx}{xx}$ into $\frac{cx}{aa+ax}$ becomes $\frac{a-x}{x}$, into $\frac{c}{1}$, or $\frac{ac}{x}-c$. And 28 divided by $\frac{7}{3}$ becomes 4 divided by $\frac{1}{3}$, or 12.

## *Of the Invention of Diviſors.*

TO this Head may be referr'd the Invention of Diviſors, by which any Quantity may be divided. If it be a ſimple Quantity, divide it by its leaſt Diviſor, and the Quotient by its leaſt Diviſor, till there remain an indiviſible Quotient, and you will have all the prime Diviſors of [that] Quantity. Then multiply together each Pair of theſe Diviſors, each ternary [or three] of them, each quaternary, *&c.* and you will alſo have all the compounded Diviſors. As, if all the Diviſors of the Number 60 are requir'd, divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and there will remain the indiviſible Quotient 5. Therefore the prime Diviſors are 1, 2, 2, 3, 5; thoſe compos'd of the Pairs 4, 6, 10, 15; of the Ternaries 12, 20, 30; and of all of them 60. Again, If all the Diviſors of the Quantity $21abb$ are deſir'd, divide it by 3, and the Quotient $7abb$ by 7, and the Quotient $abb$ by $a$, and the Quotient $bb$ by $b$, and there will remain the prime Quotient $b$. Therefore the prime Diviſors are 1, 3, 7, $a$, $b$, $b$; and thoſe compos'd of the Pairs 21, $3a$, $3b$, $7a$, $7b$, $ab$, $bb$; thoſe compos'd of the Ternaries $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, $abb$; and thoſe of the Quaternaries $21ab$, $21bb$, $3abb$, $7abb$; that of the Quinaries $21abb$. After the ſame Way all the Diviſors of $2abb-6aac$ are 1, 2, $a$, $bb-3ac$, $2a$, $2bb-6ac$, $abb-3aac$, $2abb-6aac$.

If after a Quantity is divided by all its ſimple Diviſors, it remains [ſtill] compounded, and you ſuſpect it has ſome compounded Diviſor, [order it or] diſpoſe it according to the Dimenſions of any of the Letters in it, and in the Room of that Letter ſubſtitute ſucceſſively three or more Terms of this Arithmetical Progreſſion, *viz.* 3, 2, 1, 0, $-1$, $-2$, and ſet the reſulting Terms together with all their Diviſors, by the correſponding Terms of the Progreſſion, ſetting down alſo the Signs of the Diviſors, both Affirmative and Negative. Then ſet alſo down the Arithmetical Progreſſions which run thro' the Diviſors of all the Numbers proceeding from the greater Terms to the leſs, in the Order that the Terms of the Progreſſion 3, 2, 1, 0, $-1$, $-2$, proceed, and whoſe Terms differ either by Unity, or by ſome Number which divides the higheſt Term of the Quantity propos'd. If any Progreſſion of this kind occurs, that Term of it which ſtands in the ſame Line with the Term 0 of the firſt Progreſſion, divided by the Difference of the Terms, will compoſe the Quantity by which you are to attempt the Diviſion.

As if the Quantity be $x^3 - xx - 10x + 6$, by ſubſtituting, one by one, the Terms of this Progreſſion 1. 0. $-1$, for $x$, there will ariſe the Numbers $-4$, 6, $+14$, which, together with all their Diviſors, I place right againſt the Terms of the Progreſſion 1. 0. $-1$. after this Manner:

| | | | |
|---|---|---|---|
| 1 | 4 | 1. 2. 4. | $+4$. |
| 0 | 6 | 1. 2. 3. 6 | $+3$. |
| $-1$ | 14 | 1. 2. 7. 14 | $+2$. |

Then, becauſe the higheſt Term $x^3$ is diviſible by no Number but Unity, I ſeek among the Diviſors a Progreſſion whoſe Terms differ by Unity, and (proceeding from the higheſt to the loweſt) decreaſe as the Terms of the lateral Progreſſion 1. 0. $-1$. And I find only one Progreſſion of this Sort, *viz.* 4. 3. 2. whoſe Term therefore $+3$ I chuſe, which ſtands in the ſame Line with the Term 0 of the firſt Progreſſion 1. 0. $-1$. and I attempt the Diviſion by $x + 3$, and [find] it ſucceeds, there coming out $xx - 4x + 2$.

Again, if the Quantity be $6y^4 - y^3 - 21yy + 3y + 20$, for $y$ I ſubſtitute ſucceſſively 1. 0. $-1$. and the reſulting Numbers 7. 20. 9. with all their Diviſors, I place by them as follows:

| | | | |
|---|---|---|---|
| 1 | 7 | 1. 7. | 7. |
| 0 | 20 | 1. 2. 4. 5. 10. 20 | 4. |
| $-1$ | 9 | 1. 3. 9 | 1. |

And among the Diviſors I perceive there is this decreaſing Arithmetical Progreſſion 7. 4. 1. The Difference of the Terms of this Progreſſion, *viz.* 3. divides the higheſt Term of the Quantity $6y^4$. Wherefore I adjoin the Term $+4$, which ſtands [in the Row] oppoſite to the Term 0, divided by the Difference of the Terms, *viz.* 3, and I attempt the Diviſion by $y+\frac{4}{3}$, or, which is the ſame Thing, by $3y+4$, and the Buſineſs ſucceeds, there coming out $2y^3-3yy-3y+5$.

And ſo, if the Quantity be $24a^5-50a^4+49a^3-140a^2+64a+30$, the Operation will be as follows:

| | | | |
|---|---|---|---|
| 2 | 42 | 1.2.3.6.7.14.21.42 | + 3. + 3. + 7. |
| 1 | 23 | 1.23. | + 1. — 1. + 1. |
| 0 | 30 | 1.2.3.5.6.10.15.30 | — 1. — 5. — 5. |
| — 1 | 297 | 1.3.9.11.27.33.99.297 | — 3. — 9. — 11. |

Here are three Progreſſions, whoſe Terms $-1.-5.-5.$ divided by the Differences of the Terms 2, 4, 6, give three Diviſors to be try'd $a-\frac{1}{2}$, $a-\frac{5}{4}$, and $a-\frac{5}{6}$. And the Diviſion by the laſt Diviſor $a-\frac{5}{6}$, or $6a-5$, ſucceeds, there coming out $4a^4-5a^3+4aa-20a-6$.

If no Diviſor occur by this Method, or none that divides the Quantity propos'd, we are to conclude, that that Quantity does not admit a Diviſor of one Dimenſion. But perhaps it may, if it be a Quantity of more than three Dimenſions, admit a Diviſor of two Dimenſions. And if ſo, that Diviſor will be found by this Method. Subſtitute in that Quantity for the Letter [or Species] as before, four or more Terms of this Progreſſion $3, 2, 1, 0. -1. -2. -3.$ Add and ſubtract ſingly all the Diviſors of the Numbers that reſult to or from the Squares of the correſpondent Terms of that Progreſſion, multiply'd into ſome Numeral Diviſor of the higheſt Term of the Quantity propos'd, and place right againſt the Progreſſion the Sums and Differences. Then note all the collateral Progreſſions which run thro' thoſe Sums and Difference. Then ſuppoſe $\mp C$ to be a Term of ſuch a prime Progreſſion, and $\mp B$ the Difference which ariſes by ſubducting $\mp C$ from the next ſuperior Term which ſtands againſt the Term 1 of the firſt Progreſſion, and A to be the aforeſaid Numeral Diviſor of the higheſt Term, and $l$ [to be] a Letter which is in the propos'd Quantity, then $All \pm Bl \pm C$ will be the Diviſor to be try'd.

Thus ſuppoſe the propos'd Quantity to be $x^4-x^3-5xx+12x-6$, for $x$ I write ſucceſſively $3, 2, 1, 0, -1$, and the Numbers that come out $39.\ 6.\ 1.\ -6.\ -21.\ -26.$ I diſpoſe [or place] together with their Diviſors in another Column in the ſame Line with them, and I add and ſubtract the

Diviſors to and from the Squares of the Terms of the firſt Progreſſion, multiply'd by the Numeral Diviſor of the Term $x^4$, which is Unity, *viz.* to and from the Terms 9.4.1.0.1.4, and I diſpoſe likewiſe the Sums and Differences on the Side. Then I write, as follows, the Progreſſions which occur among the ſame. Then I make Uſe of the Terms of theſe Progreſſions 2 and — 3, which ſtand oppoſite to the Term 0 in that Progreſſion which is in the firſt Column, ſucceſſively

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 39 | 1. 3. 13. 39 | 9 | —30. —4. 6. 8. 10. 12. 22. 48 | —4. | 6 |
| 2 | 6 | 1. 2. 3. 6 | 4 | —2. 1. 2. 3. 5. 6. 7. 10. | —2. | 3 |
| 1 | 1 | 1. | 1 | 0. 2. | 0. | 0 |
| 0 | 6 | 1. 2. 3. 6 | 0 | —6. —3. —2. —1. 1. 2. 3. 6 | 2 | —3 |
| —1 | 21 | 1. 3. 7. 21 | 1 | —20. —6. —2. 0. 2. 4. 8. 22 | 4 | —6 |
| —2 | 26 | 1. 2. 13. 26 | 4 | —22. — 9. 2. 3. 5. 6. 17. 30 | 6 | —9 |

for $\mp C$, and I make Uſe of the Differences that ariſe by ſubtracting theſe Terms from the ſuperior Terms 0 and 0, *viz.* — 2 and + 3 reſpectively for $\mp B$. Alſo Unity for A; and $x$ for $l$. And ſo in the Room of $All \pm Bl \pm C$, I have theſe two Diviſors to try, *viz.* $xx + 2x - 2$, and $xx - 3x + 3$, by both of which the Buſineſs ſucceeds.

Again, if the Quantity $3y^5 - 6y^4 + y^3 - 8yy - 14y + 14$ be propos'd, the Operation will be as follows: Firſt, I attempt the Buſineſs by adding and ſubtracting to and from the Squares of the Terms of the Progreſſion 1.0.—1, making Uſe of 1 firſt, but the Buſineſs does not ſucceed. Wherefore, in the room of A, I make Uſe of 3, the other Diviſor of the higheſt Term; and theſe Squares being multiply'd by 3, I add and ſubtract the Diviſors to and from the Products, *viz.* 12.3.0.3, and I find theſe two Progreſſions in the reſulting Terms, —7.—7.—7.—7, and 11.5.—1.—7. For Expedition ſake, I had neglected the Diviſors of the outermoſt Terms 170 and 190. Wherefore, the Progreſſions being continu'd upwards and downwards, I take the next Terms, *viz.* —7 and 17 at the Top, and —7 and —13 at Bottom, and I try if theſe being ſubducted from the Num-

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 170 | | 27 | | —7. | 17 |
| 2 | 38 | 1.2.19.38 | 12 | —16 —7.10.11.13.14.31.50 | —7. | —11 |
| 1 | 10 | 1.2. 5.10 | 3 | —7. —2. 1. 2. 4. 5. 8. 13 | —7. | 5 |
| 0 | 14 | 1.2. 7.14 | 0 | —14 —7. —2. —1.1.2.7.14 | —7. | —1 |
| —1 | 10 | 1.2. 5.10 | 3 | —7. —2. 1. 2. 4. 5. 8. 13 | —7. | —7 |
| —2 | 190 | | 12 | | —7. | —13 |

bers 27 and 12, which ſtand againſt them in the 4th Column, [their] Differences divide thoſe [Numbers] 170 and 190, which ſtand againſt them in the ſecond Column. And the Difference between 27 and $-7$, that is, 34, divides 170; and the Difference of 12 and $-7$, that is, 19, divides 190. Alſo the Difference between 12 and 13, that is, 10, divides 170, but the Difference between 27 and 17, that is, 25, does not divide 190. Wherefore I reject the latter Progreſſion. According to the former, $\mp C$ is $-7$, and $\mp B$ is nothing; the Terms of the Progreſſion having no Difference. Wherefore the Diviſor to be try'd $All \pm Bl \pm C$ will be $3yy + 7$. And the Diviſion ſucceeds, there coming out $y^3 - 2yy - 2y + 2$.

If after this Way, there can be found no Diviſor which ſucceeds, we are to conclude, that the propos'd Quantity will not admit of a Diviſor of two Dimenſions. The ſame Method may be extended to the Invention of Diviſors of more Dimenſions, by ſeeking in the aforeſaid Terms and Differences, not Arithmetical Progreſſions, but ſome others, the firſt, ſecond, and third Differences of whoſe Terms are in Arithmetical Progreſſion: But the Learner ought not to be detain'd about them.

Where there are two Letters in the propos'd Quantity, and all its Terms aſcend to equally high Dimenſions; put Unity for one of thoſe Letters, then, by the preceding Rules, ſeek a Diviſor, and compleat the deficient Dimenſions of this Diviſor, by reſtoring that Letter for Unity. As if the Quantity be $6y^4 - cy^3 - 21ccyy + 3c^3y + 20c^4$, where all the Terms are of four Dimenſions, for $c$ I put 1, and the Quantity becomes $6y^4 - y^3 - 21yy + 3y + 20$, whoſe Diviſor, as above, is $3y + 4$; and having compleated the deficient Dimenſion of the laſt Term by a [correſpondent] Dimenſion of $c$, you have $3y + 4c$ [for] the Diviſor ſought. So, if the Quantity be $x^4 - bx^3 - 5bbxx + 12b^3x - 6b^4$, putting 1 for $b$, and having found $xx + 2x - 2$ the Diviſor of the reſulting Quantity $x^4 - x^3 - 5xx + 12x - 6$, I compleat its deficient Dimenſions by [reſpective] Dimenſions of $b$, and ſo I have $xx + 2bx + 2bb$ the Diviſor ſought.

Where there are three or more Letters in the Quantity propos'd, and all its Terms aſcend to the ſame Dimenſions, the Diviſor may be found by the precedent Rules; but more expeditiouſly after this Way: Seek all the Diviſors of all the Terms in which ſome [one] of the Letters is not, and alſo of all the Terms in which ſome other of the Letters is not; as alſo of all the Terms in which a third, fourth, and fifth Letter is not, if there are ſo many Letters; and ſo run over all the Letters: And in the ſame Line with thoſe Letters place the Diviſors reſpectively. Then ſee if in any Series of Diviſors going through all the Letters, all the Parts involving, only one Letter can be as often found as there are Letters (excepting only one) in the Quantity propos'd; and [likewiſe] if the Parts involving two Letters [may be found] as often as there are Letters (excepting two) in the Quantity propos'd. If ſo, all thoſe Parts taken together under their [proper] Signs will be the Diviſor ſought.

As if there were propos'd the Quantity $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$; the Diviſors of one Dimenſion of the Terms $8b^3 - 12bbc - 4bcc + 6c^3$, in which $x$ is not (found out by the preceding Rules) will be $2b - 3c$, and $4b - 6c$; and of the Terms $12x^3 + 9cxx + 8ccx + 6c^3$, in which $b$ is not, there will be only one Diviſor $4x + 3c$; and of the Terms $12x^3 - 14bxx - 12bbx + 8b^3$, in which there is not $c$, there will be the Diviſors $2x - b$ and $4x - 2b$. I diſpoſe theſe Diviſors in the ſame Lines with the Letters $x$, $b$, $c$, as you here ſee;

$$\begin{array}{l|l} x & 2b - 3c. \ 4b - 6c. \\ b & 4x + 3c. \\ c & 2x - b. \ 4x - 2b. \end{array}$$

Since there are three Letters. and each of the Parts of the Diviſors only involve one of the Letters, thoſe Parts ought to be found twice in the Series of Diviſors. But the Parts $4b$, $6c$, $2x$, $b$ of the Diviſors $4b - 6c$ and $2x - b$, only occur once, and are not found any where out of thoſe Diviſors whereof they are Parts. Wherefore I neglect thoſe Diviſors. There remain only three Diviſors $2b - 3c$, $4x + 3c$, and $4x - 2b$. Theſe are in the Series going through all the Letters $x$, $b$, $c$, and each of the Parts $2b$, $3c$, $4x$, are found in them twice as ought to be, and that with the ſame Signs, if only the Signs of the Diviſor $2b - 3c$ be chang'd, and in its place you write $-2b + 3c$. For you may change the Signs of any Diviſor. I take therefore all the Parts of theſe, *viz.* $2b$, $3c$, $4x$ once [apiece] under their [proper] Signs, and the Aggregate $-2b + 3c + 4x$ will be the Diviſor which was to be found. For if by this you divide the propos'd Quantity, there will come out $3xx - 2bx + 2cc - 4bb$.

Again, if the Quantity be $12x^5 - 10ax^4 - 9bx^4 - 26a^2x^3 + 12abx^3 + 6bbx^3 + 24a^3xx - 8aabxx - 8abbxx - 24b^3xx - 4a^3bx + 6aabbx - 12ab^3x + 18b^4x + 12a^4b + 32aab^3 - 12b^5$, I place the Diviſors of the Terms in which $x$ is not, by $x$; and thoſe Terms in which $a$ is not, by $a$; and thoſe in which $b$ is not, by $b$, as you here ſee. Then I perceive that all thoſe that

$$\begin{array}{c|l} x & b.\ 2b.\ 4b.\ aa + 3bb.\ 2aa + 6bb.\ 4aa + 12bb. \\ & \quad bb - 3aa.\ 2bb - 6aa.\ 4bb - 12aa. \\ a & 4xx - 3bx + 2bb.\ 12xx - 9bx + 6bb. \\ b & x.\ 2x.\ 3x - 4a.\ 6x - 8a.\ 3xx - 4ax.\ 6xx - 8ax. \\ & \quad 2xx + ax - 3aa.\ 4xx + 2ax - 6aa. \end{array}$$

are but of one Dimenſion are to be rejected, becauſe the Simple ones, $b.\ 2b.\ 4b.\ x.\ 2x$, and the Parts of the compounded ones, $3x - 4a.\ 6x - 8a$, are found but once in all the Diviſors; but there are three Letters in the propos'd Quantity, and thoſe Parts involve but one, and ſo ought to be found twice. In like Manner, the Diviſors of two Dimenſions, $aa + 3bb.\ 2aa + 6bb.\ 4aa + 12bb.\ bb - 3aa.$ and $4bb - 12aa$ I reject, becauſe their Parts $aa.\ 2aa.\ 4aa.\ bb.$ and $4bb$, involving only one Letter $a$ or $b$, are not found more than once. But the Parts $2bb$ and $6aa$ of the Diviſor $2bb - 6aa$, which is the only remaining one in the Line with $x$, and which likewiſe involve only one Letter, are found again [or twice], *viz.* the Part $2bb$ in the Diviſor $4xx - 3bx + 2bb$, and the Part $6aa$ in the Diviſor $4xx + 2ax - 6aa$. Moreover, theſe three Diviſors are in a Series ſtanding in the ſame Lines with the three Letters $x, a, b$; and all their Parts $2bb$, $6aa$, $4xx$, which involve only one Letter, are found twice in them, and that under their proper Signs; but the Parts $3bx$, $2ax$, which involve two Letters, occur but once in them. Wherefore, all the divers Parts of theſe three Diviſors, $2bb$, $6aa$, $4xx$, $3bx$, $2ax$, connected under their proper Signs, will make the Diviſors ſought, *viz.* $2bb - 6aa + 4xx - 3bx + 2ax$. I therefore divide the Quantity propos'd by this [Diviſor] and there ariſes $3x^3 - 4axx - 2aab - 6b^3$.

If all the Terms of any Quantity are not equally high, the deficient Dimenſions muſt be fill'd up by the Dimenſions of any aſſum'd Letter; then having found a Diviſor by the precedent Rules, the aſſum'd Letter is to be blotted out. As if the Quantity be $12x^3 - 14bxx + 9xx - 12bbx - 6bx + 8x + 8b^3 - 12b^2 - 4b + 6$; aſſume any Letter, as $c$, and fill up the Dimenſions of the Quantity propos'd by its Dimenſions, after this Manner, $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$. Then having found out its Diviſor $4x - 2b + 3c$, blot out $c$, and you'll have the Diviſor requir'd, *viz.* $4x - 2b + 3$.

Sometimes Diviſors may be found more eaſily than by theſe Rules. As if ſome Letter in the propos'd Quantity be of only one Dimenſion, you may ſeek for the greateſt common Diviſor of the Terms in which that Letter is found, and of the remaining Terms in which it is not found; for that Diviſor will divide the whole. And if there is no ſuch common Diviſor, there will be no Diviſor of the whole. For Example, if there be propos'd the Quantity $x^4 - 3ax^3 - 8aaxx + 18a^3x - cx^3 + acxx + 8aacx - 6a^3c - 8a^4$, let there be ſought the common Diviſor of the Terms $-cx^3 + acxx + 8aacx - 6a^3c$, in which $c$ is only of one Dimenſion, and of the remaining Terms $x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4$, and that Diviſor, *viz.* $xx + 2ax - 2aa$, will divide the whole Quantity.

But the greateſt common Diviſor of two Numbers, if it is not known [or does not appear] at firſt Sight, it is found by a perpetual Subtraction of the leſs from the greater, and of the Remainder from the [laſt Quantity] ſubtracted; and that will be the ſought Diviſor, which leaves nothing. Thus, to find the greateſt common Diviſor of the Numbers 203 and 667, ſubtract thrice 203 from 667, and the Remainder 58 thrice from 203, and the Remainder 29 twice from 58, and there will remain nothing; which ſhews, that 29 is the Diviſor ſought.

After the ſame Manner the common Diviſor in Species, when it is compounded, is found, by ſubtracting either Quantity, or its Multiple, from the other; if thoſe Quantities and the Remainder be order'd [or rang'd] according to the Dimenſions of any Letter, as is ſhewn in Diviſion, and be each Time manag'd by dividing them by all their Diviſors, which are either Simple, or divide each of its

Terms as if it were a Simple one. Thus, to find the greateſt common Diviſor of the Numerator and Denominator of this Fraction $\frac{x^4-3ax^3-8aaxx+18a^3x-8a^4}{x^3-axx-8aax+6a^3}$, multiply the Denominator by $x$, that its firſt Term may become the ſame with the firſt Term of the Numerator. Then ſubtract it, and there will remain $-2ax^3+12a^3x-8a^4$, which being rightly order'd by dividing by $-2a$, it becomes $x^3-6a^2x+4a^3$. Subtract this from the Denominator, and there will remain $-axx-2aax+2a^3$; which again divided by $-a$ becomes $xx+2ax-2aa$. Multiply this by $x$, that its firſt Term may become the ſame with the firſt Term of the laſt ſubtracted Quantity $x^3-6aax+4a^3$, from which it is to be [likewiſe] ſubtracted, and there will remain $-2axx-4aax+4a^3$, which divided by $-2a$, becomes alſo $xx+2ax-2aa$. And ſince this is the ſame with the former Remainder, and conſequently being ſubtracted from it, will leave nothing, it will be the Diviſor ſought; by which the propos'd Fraction, by dividing both the Numerator and Denominator by it, may be reduc'd to a more Simple one, *viz.* to $\frac{xx-5ax+4aa}{x-3a}$.

And ſo, if you have the Fraction

$$\frac{6a^5+15a^4b-4a^3cc-10aabcc}{9a^3b-27aabc-6abcc+18bc^3},$$

its Terms muſt be firſt abbreviated, by dividing the Numerator by $aa$, and the Denominator by $3b$: Then ſubtracting twice $3a^3-9aac-2acc+6c^3$ from $6a^3+15aab-4acc-10bcc$, there will remain $\begin{matrix}15b\\+18c\end{matrix}\,aa\,\begin{matrix}-10bcc\\-12c^3\end{matrix}$. Which being order'd, by dividing each Term by $5b+6c$ after the ſame Way as if $5b+6c$ was a ſimple Quantity, it becomes $3aa-2cc$. This being multiply'd by $a$, ſubtract it from $3a^3-9aac-2acc+6c^3$, and there will remain $-9aac+6c^3$, which being again order'd by a Diviſion by $-3c$, becomes alſo $3aa-2cc$, as before. Wherefore $3aa-2cc$ is the Diviſor ſought. Which being found, divide by it the Parts of the propos'd Fraction, and you'll have $\frac{2a^3+5aab}{3ab-9bc}$.

Now, if a common Diviſor cannot be found after this Way, it is certain there is none at all; unleſs, perhaps, it be one of the Terms that abbreviate the Numerator and Denominator of the Fraction: As, if you have the Fraction $\frac{aadd-ccdd-aacc+c^4}{4aad-4acd-2acc+2c^3}$, and ſo diſpoſe its Terms, according to the Dimenſions of the $d$, that the Numerator may become $\begin{matrix}aa\\-cc\end{matrix}dd\begin{matrix}-aacc\\+c^4\end{matrix}$, and the Denominator $\begin{matrix}4aa\\-4ac\end{matrix}d\begin{matrix}-2acc\\+2c^3\end{matrix}$. This muſt firſt be abbreviated, by dividing each Term of the Numerator by $aa-cc$, and each of the Denominator by $2a-2c$, juſt as if $aa-cc$ and $2a-2c$ were ſimple Quantities; and ſo, in Room of the Numerator there will come out $dd-cc$, and in Room of the Denominator $2ad-cc$, from which, thus prepar'd, no common Diviſor can be obtain'd. But, out of the Terms $aa-cc$ and $2a-2c$, by which both the Numerator and Denominator are abbreviated, there comes out a Diviſor, *viz.* $a-c$, by which the Fraction may be reduc'd to this, *viz.* $\frac{aad+cdd-acc-c^3}{4ad-2cc}$. Now, if neither the Terms $aa-cc$ and $2a-2c$ had not had a common Diviſor, the propoſ'd Fraction would have been irreducible.

And this is a general Method of finding common Diviſors; but moſt commonly they are more expeditiouſly found by ſeeking all the prime Diviſors of either of the Quantities, that is, ſuch as cannot be divided by others, and then by trying if any of them will divide the other without a Remainder. Thus, to reduce $\frac{a^3-aab+abb-b^3}{aa-ab}$ to the leaſt Terms, you muſt find the Diviſors of the Quantity $aa-ab$, viz. $a$ and $a-b$; then you muſt try whether either $a$, or $a-b$, will alſo divide $a^3-aab+abb-b^3$ without any Remainder.

Of

## *Of the* REDUCTION *of* FRACTIONS *to a common Denominator.*

FRACTIONS are reduc'd to a common Denominator by multiplying the Terms of each by the Denominator of the other. Thus, having $\frac{a}{b}$ and $\frac{c}{d}$, multiply the Terms of one $\frac{a}{b}$ by $d$, and alſo the Terms of the other $\frac{c}{d}$ by $b$, and they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$, whereof the common Denominator is $bd$. And thus $a$ and $\frac{ab}{c}$, or $\frac{a}{1}$ and $\frac{ab}{c}$ become $\frac{ac}{c}$ and $\frac{ab}{c}$. But where the Denominators have a common Diviſor, it is ſufficient to multiply them alternately by the Quotients. Thus the Fraction $\frac{a^3}{bc}$ and $\frac{a^3}{bd}$ are reduc'd to theſe $\frac{a^3d}{bcd}$ and $\frac{a^3c}{bcd}$, by multiplying alternately by the Quotients $c$ and $d$, ariſing by the Diviſion of the Denominators by the common Diviſor $b$.

This Reduction is moſtly of Uſe in the Addition and Subſtraction of Fractions, which, if they have different Denominators, muſt be firſt reduc'd to the ſame [Denominator] before they can be added. Thus $\frac{a}{b}+\frac{c}{d}$ by Reduction becomes $\frac{ad}{bd}+\frac{bc}{bd}$, or $\frac{ad+bc}{bd}$, and $a+\frac{ab}{c}$ becomes $\frac{ac+ab}{c}$. And $\frac{a^3}{bc}-\frac{a^3}{bd}$ becomes $\frac{a^3d-a^3c}{bcd}$, or $\frac{d-c}{bcd}a^3$. And $\frac{c^4+x^4}{cc-xx}-cc-xx$ becomes $\frac{2x^4}{cc-xx}$. And ſo $\frac{2}{3}+\frac{5}{7}$ becomes $\frac{14}{21}+\frac{15}{21}$, or $\frac{14+15}{21}$, that is, $\frac{29}{21}$. And $\frac{11}{6}-\frac{3}{4}$ becomes $\frac{22}{12}-\frac{9}{12}$, or $\frac{13}{12}$. And $\frac{3}{4}-\frac{5}{12}$

becomes $\frac{9}{12} - \frac{5}{12}$, or $\frac{4}{12}$, that is $\frac{1}{3}$. And $3\frac{4}{7}$; or $\frac{3}{1} + \frac{4}{7}$ becomes $\frac{21}{7} + \frac{4}{7}$, or $\frac{25}{7}$. And $25\frac{1}{2}$ becomes $\frac{51}{2}$.

Where there are more Fractions [than two] they are to be added gradually. Thus, having $\frac{aa}{x} - a + \frac{2xx}{3a} - \frac{ax}{a-x}$; from $\frac{aa}{x}$ take $a$, and there will remain $\frac{aa-ax}{x}$; to this add $\frac{2xx}{3a}$, and there will come out $\frac{3a^3 - 3aax + 2x^3}{3ax}$, from whence, laſtly, take away $\frac{ax}{a-x}$, and there will remain $\frac{3a^4 - 6a^3x + 2ax^3 - 2x^4}{3aax - 3axx}$. And ſo if you have $3\frac{4}{7} - \frac{2}{3}$, firſt, you are to find the Aggregate of $3\frac{4}{7}$, *viz.* $\frac{25}{7}$, and then to take from it $\frac{2}{3}$, and there will remain $\frac{61}{21}$.

## *Of the* REDUCTION *of* RADICAL [*Quantities*] *to their leaſt Terms.*

A Radical [Quantity,] where the Root of the whole cannot be extracted, is perform'd by extracting the Root of ſome Diviſor [of it]. Thus $\sqrt{aabc}$, by extracting the Root of the Diviſor $aa$, becomes $a\sqrt{bc}$. And $\sqrt{48}$, by extracting the Root of the Diviſor 16, becomes $4\sqrt{3}$. And $\sqrt{48aabc}$, by extracting the Root of the Diviſor $16aa$, becomes $4a\sqrt{3bc}$. And $\sqrt{\frac{a^3b - 4aabb + 4ab^3}{cc}}$, by extracting the Root of its Diviſor $\frac{aa - 4ab + 4bb}{cc}$, becomes $\frac{a-2b}{c}\sqrt{ab}$. And $\sqrt{\frac{aaoomm}{ppzz} + \frac{4aammmm}{pzz}}$, by extracting the

Root of the Diviſor $\frac{aamm}{ppzz}$, becomes $\frac{am}{pz}\sqrt{oo+4mp}$ And $6\sqrt{\frac{75}{98}}$, by extracting the Root of the Diviſor $\frac{25}{49}$, becomes $\frac{30}{7}\sqrt{\frac{3}{2}}$, or $\frac{30}{7}\sqrt{\frac{6}{4}}$, and by yet extracting the Root of the Denominator, it becomes $\frac{15}{7}\sqrt{6}$. And ſo $a\sqrt{\frac{b}{a}}$, or $a\sqrt{\frac{ab}{aa}}$, by extracting the Root of the Denominator, becomes $\sqrt{ab}$. And $\sqrt[3]{8a^3b+16a^4}$, by extracting the Cube Root of its Diviſor $8a^3$, becomes $2a\sqrt[3]{b+2a}$. And not unlike [this] $\sqrt[4]{a^3x}$, by extracting the Square Root of its Diviſor $aa$, becomes $\sqrt{a}$ into $\sqrt[4]{ax}$, or by extracting the Biquadratick Root of the Diviſor $a^4$, it becomes $a\sqrt[4]{\frac{x}{a}}$. And ſo $\sqrt[6]{a^7x^5}$ is chang'd into $a\sqrt[6]{ax^5}$, or into $ax\sqrt[6]{\frac{a}{x}}$, or into $\sqrt{ax}\times\sqrt[3]{aax}$.

Moreover, this Reduction is not only of Uſe for abbreviating of Radical Quantities, but alſo for their Addition and Subtraction, if they agree in their Roots when they are reduc'd to the moſt ſimple Form; for then they may be added, which otherwiſe they cannot. Thus, $\sqrt{48}+\sqrt{75}$ by Reduction becomes $4\sqrt{3}+5\sqrt{3}$, that is, $9\sqrt{3}$. And $\sqrt{48}-\sqrt{\frac{16}{27}}$ by Reduction becomes $4\sqrt{3}-\frac{4}{9}\sqrt{3}$, that is, $\frac{32}{9}\sqrt{3}$. And thus, $\sqrt{\frac{4ab^3}{cc}}+\sqrt{\frac{a^3b-4aabb+4ab^3}{cc}}$, by extracting what is Rational in it, becomes $\frac{2b}{c}\sqrt{ab}+\frac{a-2b}{c}\sqrt{ab}$, that is, $\frac{a}{c}\sqrt{ab}$. And $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}$ becomes $2a\sqrt[3]{b+2a}-b\sqrt[3]{b+2a}$, that is, $2a-b\sqrt[3]{b+2a}$.

Of

## *Of the* REDUCTION *of* RADICAL [*Quantities*] *to the ſame Denomination.*

WHEN you are to multiply or divide Radicals of a different Denomination, you muſt [firſt] reduce them to the ſame Denomination, by prefixing that Radical Sign whoſe Index is the leaſt Number, which their Indices divide without a Remainder, and by multiplying the Quantities under the Signs ſo many times, excepting one, as that Index is become greater. For ſo $\sqrt{ax}\sqrt[3]{aax}$ becomes $\sqrt[6]{a^3x^3}$ into $\sqrt[6]{a^4xx}$, that is, $\sqrt[6]{a^7x^5}$. And $\sqrt{a}$ into $\sqrt[4]{ax}$ becomes $\sqrt[4]{aa}$ into $\sqrt[4]{ax}$, that is, $\sqrt[4]{a^3x}$. And $\sqrt{6}$ into $\sqrt[4]{\frac{5}{6}}$ becomes $\sqrt[4]{36}$ into $\sqrt[4]{\frac{5}{6}}$, that is, $\sqrt[4]{30}$. By the ſame Reaſon, $a\sqrt{bc}$ becomes $\sqrt{aa}$ into $\sqrt{bc}$, that is, $\sqrt{aabc}$. And $4a\sqrt{3bc}$ becomes $\sqrt{16aa}$ into $\sqrt{3bc}$, that is $\sqrt{48aabc}$. And $2a\sqrt[3]{b+2a}$ becomes $\sqrt[3]{8a^3}$ into $\sqrt[3]{b+2a}$, that is, $\sqrt[3]{8a^3b+16a^4}$. And ſo $\frac{\sqrt{ac}}{b}$ becomes $\frac{\sqrt{ac}}{\sqrt{bb}}$, or $\sqrt{\frac{ac}{bb}}$. And $\frac{6abb}{\sqrt{18ab^3}}$ becomes $\frac{\sqrt{36aab^4}}{\sqrt{18ab^3}}$, or $\sqrt{2ab}$. And ſo in others.

## *Of the* REDUCTION *of* RADICALS *to more ſimple Radicals, by the Extraction of Roots.*

THE Roots of Quantities, which are compos'd of Integers and Radical Quadraticks, extract thus: Let A denote the greater Part of any Quantity, and B the leſſer Part; and $\frac{A+\sqrt{AA-BB}}{2}$ will be the Square of the greater Part of the Root; and $\frac{A-\sqrt{AA-BB}}{2}$ will be the Square of the leſſer Part, which is to be joyn'd to the greater Part with the Sign of B. As if the Quantity be $3+\sqrt{8}$, by writing 3 for A, and $\sqrt{8}$ for B, $\sqrt{AA-BB}=1$, and thence the Square of the greater Part of the Root $\frac{3+1}{2}$, that is, 2, and the Square of the leſs $\frac{3-1}{2}$, that is, 1. Therefore the Root is $1+\sqrt{2}$. Again, if you are to extract the Root of $\sqrt{32}-\sqrt{24}$, by putting $\sqrt{32}$ for A, and $\sqrt{24}$ for B, $\sqrt{AA-BB}$ will $=\sqrt{8}$, and thence $\frac{\sqrt{32}+\sqrt{8}}{2}$, and $\frac{\sqrt{32}-\sqrt{8}}{2}$, that is, $3\sqrt{2}$ and $\sqrt{2}$ will be the Squares of the Parts of the Root. The Root therefore is $\sqrt[4]{18}-\sqrt[4]{2}$. After the ſame manner, if, out of $aa+2x\sqrt{aa-xx}$ you are to extract the Root, for A write $aa$, and for B $2x\sqrt{aa-xx}$, and $AA-BB$ will $=a^4-4aaxx+4x^4$, the Root whereof is $aa-2xx$. Whence the Square of one Part of the Root will be $aa-xx$, and that of the other $xx$; and ſo the Root [will be] $x+\sqrt{aa-xx}$. Again, if you have $aa+5ax-2a\sqrt{ax+4xx}$, by writing $aa+5ax$ for A, and $2a\sqrt{ax+4xx}$ for B, $AA-BB$ will $=a^4+6a^3x+9aaxx$, whoſe Root is $aa+3ax$. Whence the Square of the greater Part of the Root will be $aa+4ax$, and that of the leſſer Part $ax$, and the Root $\sqrt{aa+4ax}-\sqrt{ax}$. Laſtly, if you have $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$, putting $6+\sqrt{8}=A$, and $-\sqrt{12}-\sqrt{24}=B$, $AA-BB=8$; whence the greater Part of the Root is $\sqrt{3+\sqrt{8}}$, that is as above $1+\sqrt{2}$, and the leſſer Part $\sqrt{3}$, and conſequently the Root it ſelf $1+\sqrt{2}-\sqrt{3}$. But where there are more of this ſort of Radical Terms, the Parts of the Root may be ſooner found, by dividing the Product of any two of the Radicals by ſome third Radical, which [ſhall] produce a Rational and Integer Quotient. For the Root of that Quotient will be double of the Part of the Root ſought. As in the laſt Example, $\frac{\sqrt{8}\times\sqrt{12}}{\sqrt{24}}=2$. $\frac{\sqrt{8}\times\sqrt{24}}{\sqrt{12}}=4$. And $\frac{\sqrt{12}\times\sqrt{24}}{\sqrt{8}}=6$. Therefore the Parts of the Root are 1, $\sqrt{2}$, $\sqrt{3}$ as above.

There is alſo a Rule of extracting higher Roots out of Numeral Quantities [conſiſting] of two Parts, whoſe Squares are commenſurable. Let there be the Quantity $A \pm B$. And its greater Part A. And the Index of the Root to be extracted $c$. Seek the leaſt Number N, whoſe Power $N^c$ is [may be] divided by $AA - BB$, without any Remainder, and let the Quotient be Q. Compute $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$ in the neareſt Integer Numbers. Let it be $r$. Divide $A\sqrt{Q}$ by the greateſt rational Diviſor. Let the Quotient be $s$, and let $\frac{r+\frac{n}{r}}{2s}$ in the next greateſt Integers be [called] $t$. And $\frac{ts \pm \sqrt{ttss-n}}{\sqrt[2c]{Q}}$ will be the Root ſought, if the Root can be extracted.

As if the Cube Root be to be extracted out of $\sqrt{968} + 25$; $AA - BB$ will $= 343$; and 7, 7, 7 will be its Diviſors; therefore $N = 7$ and $Q = 1$. Moreover, $\overline{A+B} \times \sqrt{Q}$, or $\sqrt{968} + 25$, having extracted the former Part of the Root is a little greater than 56, and its Cube Root in the neareſt Numbers is 4; therefore $r = 4$. Moreover, $A\sqrt{Q}$, or $\sqrt{968}$, by taking out whatever is Rational, becomes $22\sqrt{2}$. Therefore $\sqrt{2}$ its Radical Part is $s$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{5\frac{3}{4}}{2\sqrt{2}}$ in the neareſt Integer Numbers is 2. Therefore $t = 2$. Laſtly, $ts$ is $2\sqrt{2}$, $\sqrt{ttss-n}$ is 1, and $\sqrt[2c]{Q}$, or $\sqrt[6]{1}$, is 1. Therefore $2\sqrt{2}+1$ is the Root ſought, if it can be extracted. I try therefore by Multiplication if the Cube of $2\sqrt{2}+1$ be $\sqrt{968}+25$, and it ſucceeds.

Again, if the Cube Root is to be extracted out of $68 - \sqrt{4374}$, $AA - BB$ will be $= 250$, whoſe Diviſors are 5, 5, 5, 2. Therefore $N = 5 \times 2 = 10$, and $Q = 4$. And $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$, or $\sqrt[3]{\overline{68+\sqrt{4374}} \times 2}$ in the neareſt Integer Numbers is $7 = r$. Moreover, $A\sqrt{Q}$, or $68\sqrt{4}$, by ex-

extracting [or taking out] what is Rational, becomes $136\sqrt{1}$. Therefore $s = 1$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{7+\frac{10}{7}}{2}$ in the neareſt Integer Numbers is $4 = t$. Therefore $ts = 4$, $\sqrt{ttss - n} = \sqrt{6}$, and $\sqrt[2c]{Q} = \sqrt[6]{4}$, or $\sqrt[3]{2}$; and ſo the Root to be try'd is $\frac{4-\sqrt{6}}{\sqrt[3]{2}}$.

Again, if the fifth Root be to be extracted out of $29\sqrt{6}+41\sqrt{3}$; $AA - BB$ will be $= 3$, and conſequently $N = 3$, $Q = 81$, $r = 5$, $s = \sqrt{6}$, $t = 1$, $ts = \sqrt{6}$, $\sqrt{ttss - n} = \sqrt{3}$, and $\sqrt[2c]{Q} = \sqrt[10]{81}$, or $\sqrt[5]{9}$; and ſo the Root to be try'd is $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

But if in theſe Sorts of Operations, the Quantity be a Fraction, or its Parts have a common Diviſor, extract ſeparately the Roots of the Terms, and of the Factors. As if the Cube Root be to be extracted out of $\sqrt{242} - 12\frac{1}{2}$, this, having reduc'd its Parts to a common Denominator, will become $\frac{\sqrt{968}-25}{2}$. Then having extracted ſeparately the Cube Root of the Numerator and the Denominator, there will come out $\frac{2\sqrt{2}-1}{\sqrt[3]{2}}$. Again, if you are to extract any Root out of $\sqrt[3]{3993}+\sqrt[6]{17578125}$; divide the Parts by the common Diviſor $\sqrt[3]{3}$, and there will come out $11+\sqrt{125}$. Whence the propos'd Quantity is $\sqrt[3]{3}$ into $11+\sqrt{125}$, whoſe Root will be found by extracting ſeparately the Root of each Factor $\sqrt[3]{3}$, and $11+\sqrt{125}$.

## *Of the Form of an* ÆQUATION.

ÆQUATIONS, which are either two Ranks of Quantities, equal to one another, or one Rank taken equal to nothing, are to be conſider'd chiefly after two Ways; either as the laſt Concluſions to which you come in the Reſolution of Problems; or as Means, by the Help whereof you are to obtain [other] final Æquations. An Æquation of the former Kind is compos'd only out of one unknown Quantity involv'd with known ones, If the Problem be determin'd, and propoſes ſomething certain to be found out. But thoſe of the latter Kind involve ſeveral unknown Quantities, which, for that Reaſon, muſt be compar'd among one another, and ſo connected, that out of all there may emerge a new Æquation, in which there is only one unknown Quantity which we ſeek; [and] that Æquation muſt be transform'd moſt commonly various Ways, untill it becomes the moſt Simple that it can, and alſo like ſome of the following Degrees of them, in which $x$ denotes the Quantity ſought, according to whoſe Dimenſions the Terms, as you ſee, are order'd, [or rang'd] and $p$, $q$, $r$, $s$, [denote] any other Quantities from which, being known and determin'd, $x$ is alſo determin'd, and may be inveſtigated by Methods hereafter to be explain'd.

$x = p.$ Or, $x - p = 0.$
$xx = px + q.$ $xx - px - q = 0.$
$x^3 = px^2 + qx + r.$ $x^3 - pxx - qx - r = 0.$
$x^4 = px^3 + qx^2 + rx + s.$ $x^4 - px^3 - qx^2 - rx - s = 0.$
&c. &c.

After this Manner therefore the Terms of Æquations are to be reduc'd, [or order'd] according to the Dimenſions of the unknown Quantity, ſo that [thoſe] may be in the firſt Place, in which the unknown Quantity is of the moſt Dimenſions, as $x$, $xx$, $x^3$, $x^4$, &c. and thoſe in the ſecond Place, in which [$x$] is of the next greateſt Dimenſion, and ſo on. As to what regards the Signs, they may ſtand any how; and one or more of the intermediate Terms may be ſometimes wanting. Thus, $x^3 * - bbx + b^3 = 0$, or $x^3 = bbx - b^3$, is an Æquation of the third Degree, and

$z^4 \begin{matrix}+a\\-b\end{matrix} z^3 * \begin{matrix}+ab^3\\-b^4\end{matrix} = 0$, is an Æquation of the fourth Degree. For the Degree of an Æquation is always eſtimated by the greateſt Dimenſion of the unknown Quantity, without any Regard to the known ones, or to the intermediate Terms. But by the Defect of the intermediate Terms, the Æquation is moſt commonly render'd much more ſimple, and may be ſometimes depreſs'd to a lower Degree. For thus, $x^4 = qxx + s$ is to be reckon'd an Æquation of the ſecond Degree, becauſe it may be reſolv'd into two Æquations of the ſecond Degree. For, ſuppoſing $xx = y$, and $y$ being accordingly writ for $xx$ in that Æquation, there will come out in its ſtead $yy = qy + s$, an Æquation of the ſecond Degree; by the Help whereof when $y$ is found, the Æquation $xx = y$ alſo of the ſecond Degree, will give $x$.

And theſe are the Concluſions to which Problems are to be brought. But before I go upon their Reſolution, it will be neceſſary to ſhew the Methods of transforming and reducing Æquations into Order, and the Methods of finding the final Æquations. I ſhall comprize the Reduction of a Simple Æquation in the following Rules.

## *Of ordering, [or managing]* &c. *a Simple* ÆQUATION.

RULE I. IF there are any Quantities that deſtroy one another, or may be joyn'd into one by Addition or Subtraction, the Terms are that Way to be diminiſh'd [or reduc'd]. As if you have $5b - 3a + 2x = 5a + 3x$, take from each Side $2x$, and add $3a$, and there will come out $5b = 8a + x$. And thus, $\frac{2ab + bx}{a} - 2b = a + b$, by ſtriking out the equivalent Quantities $\frac{2ab}{a} - b = b$, becomes $\frac{bx}{a} = a$.

To this Rule may alſo be referr'd the Ordering [or Management] of the Terms of an Æquation, which is uſually perform'd by the Tranſpoſition of the Members to the contrary Sides under the contrary Sign. As if you had the Æquation $5b = 8a + x$, you are to find $x$; take from each

Side $8a$; or, which is the ſame Thing, tranſpoſe $8a$ to the contrary Side with its Sign chang'd, and there will come out $5b - 8a = x$. After the ſame Way, if you have $aa - 3ay = ab - bb + by$, and you are to find $y$; tranſpoſe $-3ay$ and $ab - bb$, ſo that there may be the Terms multiply'd by $y$ on the one Side, and the other Terms on the other Side, and there will come out $aa - ab + bb = 3ay + by$, whence you'll have $y$ by the fifth Rule following, *viz.* by dividing each Part by $3a + b$, for there will come out $\frac{aa - ab + bb}{3a + b} = y$. And thus the Æquation $abx + a^3 - aax = abb - 2abx - x^3$, by due ordering and tranſpoſition becomes $x^3 = \begin{matrix} aa \\ -3ab \end{matrix} x \begin{matrix} -a^3 \\ +abb \end{matrix}$, or $x^3 \begin{matrix} -aa \\ +3ab \end{matrix} x \begin{matrix} +a^3 \\ -abb \end{matrix} = 0$.

RULE II. If there is any Quantity by which all the Terms of the Æquation are multiply'd, all of them muſt be divided by that Quantity; or, if all are divided by the ſame Quantity, all muſt be multiply'd by it too. Thus, having $15bb = 24ab + 3bx$, divide all the Terms by $b$, and you'll have $15b = 24a + 3x$; then by 3, and you'll have $5b = 8a + x$; or, having $\frac{b^3}{ac} - \frac{bbx}{cc} = \frac{xx}{c}$, multiply all by $c$, and there comes out $\frac{b^3}{a} - \frac{bbx}{c} = xx$.

RULE III. If there be any irreducible Fraction, in whoſe Denominator there is found the Letter [unknown], according to whoſe Dimenſions the [whole] Æquation is to be order'd [or rang'd], all the Terms of the Æquation muſt be multiply'd by that Denominator, or by ſome Diviſor of it. As if the Æquation $\frac{ax}{a - x} + b = x$ be to be order'd [or rang'd] according to $x$, multiply all its Terms by $a - x$ the Denominator of the Fraction $\frac{ax}{a - x}$, and there comes out $ax + ab - bx = ax - xx$, or $ab - bx = -xx$, and tranſpoſing each Part [you'll have] $xx = bx - ab$. And ſo if you have $\frac{a^3 - aab}{2cy - cc} = y - c$, and the Terms are to be order'd [or rang'd] according to [the Dimenſions of] $y$, multiply them by the Denominator $2cy - cc$, or, at leaſt, by its Diviſor $2y - c$, that $y$ may vaniſh in the Denominator, and there will come out $\frac{a^3 - abb}{c} = 2yy - 3cy + cc$, and by farther ordering $\frac{a^3 - abb}{c} - cc + 3cy = 2yy$. After the ſame manner $\frac{aa}{x} - a = x$, by being multiply'd by $x$, becomes $aa - ax = xx$, and $\frac{aabb}{cxx} = \frac{xx}{a + b - x}$, and multiplying firſt by $xx$, and then by $a + b - x$, it becomes $\frac{a^3bb + aab^3 - aabbx}{c} = x^4$.

Rule IV. If that [particular] Letter, according to whoſe Dimenſions the Æquation is to be order'd [or rang'd], be involv'd with an irreducible Surd, all the other Terms are to be tranſpos'd to the other Side, their Signs being chang'd, and each Part of the Æquation muſt be once multiply'd by it ſelf, if the Root be a Square one, or twice if it be a Cubick one, &c. Thus, to order the Æquation $\sqrt{aa - ax} + a = x$ according to the Letter $x$, tranſpoſe $a$ to the other Side, and you have $\sqrt{aa - ax} = x - a$; and having ſquar'd the Parts $aa - ax = xx - 2ax + aa$, or $0 = xx - ax$, that is, $x = a$. So alſo $\sqrt[3]{aax + 2axx - x^3} - a + x = 0$, by tranſpoſing $-a + x$, it becomes $\sqrt[3]{aax + 2axx - x^3} = a - x$, and multiplying the Parts cubically $aax + 2axx - x^3 = a^3 - 3aax + 3axx - x^3$, or $xx = 4ax - aa$. And ſo $y = \sqrt{ay + yy - a\sqrt{ay - yy}}$ having ſquar'd the Parts, becomes $yy = ay + yy - a\sqrt{ay - yy}$ and the Terms being rightly tranſpos'd [it becomes] $ay = a\sqrt{ay - yy}$, or $y = \sqrt{ay - yy}$, and the Parts being again ſquar'd $yy = ay - yy$; and laſtly, by tranſpoſing $2yy = ay$, or $2y = a$.

Rule V. The Terms, by help of the preceding Rules, being diſpos'd [or rang'd] according to the Dimenſions of ſome one of the Letters, if the higheſt Dimenſion of that Letter be multiply'd by any known Quantity, the whole Æquation muſt be divided by that Quantity. Thus, $2y = a$, by dividing by 2, becomes $y = \frac{1}{2}a$. And $\frac{bx}{a} = a$, by dividing by $\frac{b}{a}$, becomes $x = \frac{aa}{b}$. And $\begin{array}{l} 2ac \\ -cc \end{array} x^3 \begin{array}{l} +a^3 \\ +aac \end{array} xx \begin{array}{l} -2a^3c \\ +aacc \end{array} x - a^3cc = 0$, by dividing by $2ac - cc$, becomes

$$\frac{\begin{array}{l} 2ac \\ -cc \end{array} x^3 \begin{array}{l} +a^3 \\ +aac \end{array} x^2 \begin{array}{l} -2a^3c \\ +aacc \end{array} x - a^3cc}{2ac - cc} = 0, \text{ or}$$

$$x^3 \frac{+a^3 + aac}{2ac - cc} xx - aax - \frac{a^3c}{2a - c} = 0.$$

Rule VI. Sometimes the Reduction may be perform'd by dividing the Æquation by ſome compounded Quantity. For thus, $y^3 = \begin{array}{l} -2c \\ +b \end{array} yy + 3bcy - bbc$, is reduc'd to this, viz. $yy = 2cy + bc$, by transferring all the Terms to the ſame Side thus, $y^3 \begin{array}{l} +2c \\ -b \end{array} yy - 3bcy + bbc = 0$, and dividing by $y - b$, as is ſhewn in the Chapter of *Diviſion*; for there will come out $yy + 2cy - bc = 0$. But the Invention of this Sort of Diviſors is difficult, and is more fully taught elſewhere.

Rule VII. Sometimes alſo the Reduction is perform'd by Extraction of the Root out of each Part of the Æquation. As if you have $xx = \frac{1}{4}aa - bb$, having extracted the Root on both Sides, there comes out $x = \sqrt{\frac{1}{4}aa - bb}$. If you have $xx + aa = 2ax + bb$, tranſpoſe $2ax$ [to the other Side] and there will ariſe $xx - 2ax + aa = bb$, and extracting the Roots of the Parts $x - a = +$, or $-b$, or $x = a \pm b$. So alſo having $xx = ax - bb$, add on each Side $-ax + \frac{1}{4}aa$, and there comes out $xx - ax + \frac{1}{4}aa = \frac{1}{4}aa - bb$, and extracting the Root on each Side $x - \frac{1}{2}a = \pm\sqrt{\frac{1}{4}aa - bb}$, or $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb}$.

And thus univerſally if you have $xx = .px.q$, $x$ will be $= .\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp.q}$. Where $\frac{1}{2}p$ and $q$ are to be affected with the ſame Signs as $p$ and $.q$ in the former Æquation; but $\frac{1}{4}pp$ muſt be always made Affirmative. And this Example is a Rule according to which [or like to which] all Quadratick Æquations may be reduc'd to the Form of Simple ones. Therefore, having propos'd the Æquation $yy = \frac{2xxy}{a} + xx$, to extract the Root $y$, compare $\frac{2xx}{a}$ with $p$, that is, write $\frac{xx}{a}$ for $\frac{1}{2}p$, and $\frac{x^4}{aa} + xx$ for $\frac{1}{4}pp \cdot q$, and there will ariſe $y = \frac{xx}{a} + \sqrt{\frac{x^4}{aa} + xx}$, or $y = \frac{xx}{a} - \sqrt{\frac{x^4}{aa} + xx}$. After the ſame Way, the Æquation $yy = ay - 2cy + aa - cc$; by comparing $a - 2c$ with $p$, and $aa - cc$ with $q$, will give $y = \frac{1}{2}a - c \pm \sqrt{\frac{5}{4}aa - ac}$.

Moreover, the Biquadratick Æquation $x^4 = -aaxx + ab^3$, whoſe odd Terms are wanting, by help of this Rule becomes $xx = -\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}$, and extracting again the Root $x = \sqrt{-\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}}$. And ſo in others.

And theſe are the Rules for ordering one only Æquation, the Uſe whereof, when the Analyſt is ſufficiently acquainted with, ſo that he knows how to diſpoſe any propos'd Æquation according to any of the Letters contain'd in it, and to obtain the Value of that Letter if it be of one Dimenſion, or of its greateſt Power if it be of more; the Compariſon of ſeveral Æquations among one another will not be difficult to him, which I am now going to ſhew.

## *Of the Transformation of two or more* ÆQUATIONS *into one, in order to exterminate the unknown Quantities.*

WHEN in the Solution of any Problem, there are more Æquations than one to comprehend the State of the Queſtion, in each of which there are ſeveral unknown Quantities; thoſe Æquations (two by two, if there are more than two) are to be ſo connected, that one of the unknown Quantities may be made to vaniſh at each of the Operations, and ſo produce a new Æquation. Thus, having the Æquations $2x = y + 5$, and $x = y + 2$, by taking off equal Things out of equal Things, there will come out $x = 3$.

$x=3$. And you are to know, that by each Æquation one unknown Quantity may be taken away, and conſequently, when there are as many Æquations as unknown Quantities, all may at length be reduc'd into one, in which there ſhall be only one Quantity unknown. But if there be more unknown Quantities by one than there are Æquations, then there will remain in the Æquation laſt reſulting two unknown Quantities; and if there are more [unknown Quantities] by two than there are Æquations, then in the laſt reſulting Æquation there will remain three; and ſo on.

There may alſo, perhaps, two or more unknown Quantities be made to vaniſh, by only two Æquations. As if you have $ax-by=ab-az$, and $bx+by=bb+az$; then adding Equals to Equals, there will come out $ax+bx=ab+bb$, $y$ and $z$ being exterminated. But ſuch Caſes either argue ſome Fault to lie hid in the State of the Queſtion, or that the Calculation is erroneous, or not artificial enough. The Method by which one unknown Quantity may be [exterminated or] taken away by each of the Æquations, will appear by what follows.

## *The Extermination of an unknown Quantity by an Equality of its Values.*

WHEN the Quantity to be exterminated is only of one Dimenſion in both Æquations, both its Values are to be ſought by the Rules already deliver'd, and the one made equal to the other.

Thus, putting $a+x=b+y$, and $2x+y=3b$, that $y$ may be exterminated, the firſt Æquation will give $a+x-b=y$, and the ſecond will give $3b-2x=y$. Therefore $a+x-b=3b-2x$, or by [due] ordering $x=\frac{4b-a}{3}$.

And thus, $2x=y$, and $5+x=y$, give $2x=5+x$, or $x=5$.

And $ax-2by=ab$, and $xy=bb$, give $\frac{ax-ab}{2b}(=y)=\frac{bb}{x}$; and by [due] ordering [the Terms] [$xx-bx=\frac{2b^3}{a}$, or] $xx-bx-\frac{2b^3}{a}=0$.

Alſo $\frac{bbx - aby}{a} = ab + xy$, and $bx + \frac{ayy}{c} = 2aa$, by taking away $x$, give $\frac{aby + aab}{bb - ay} (= x) = \frac{2aac - ayy}{bc}$ and by Reduction $y^3 - \frac{bb}{a}yy - \frac{2aac + bbc}{a}y + bbc = 0$.

Laſtly, $x + y - z = 0$, and $ay = xz$, by taking away $z$, give $x + y\ (= z) = \frac{ay}{x}$, or $xx + xy = ay$.

The ſame is alſo perform'd by ſubtracting either of the Values of the unknown Quantities from the other, and making the Remainder equal to nothing. Thus, in the firſt of the Examples, take away $3b - 2x$ from $a + x - b$, and there will remain $a + 3x - 4b = 0$, or $x = \frac{4b - a}{3}$.

## *The Extermination of an unknown Quantity by ſubſtituting its Value for it.*

WHEN, at leaſt, in one of the Æquations, the Quantity to be exterminated is only of one Dimenſion, its Value is to be ſought in that Æquation, and then to be ſubſtituted in its Room in the other Æquation. Thus, having propos'd $xyy = b^3$, and $xx + yy = by - ax$, to exterminate $x$, the firſt will give $\frac{b^3}{yy} = x$; wherefore I ſubſtitute in the ſecond $\frac{b^3}{yy}$ in the Room of $x$, and there comes out $\frac{b^6}{y^4} + yy = by - \frac{ab^3}{yy}$, and by Reduction $y^6 - by^5 + ab^3yy + b^6 = 0$.

But having propos'd $ayy + aay = z^3$, and $yz - ay = az$, to take away $y$, the ſecond will give $y = \frac{az}{z - a}$. Wherefore for $y$ I ſubſtitute $\frac{az}{z - a}$ into the firſt, and there comes out $\frac{a^3zz}{zz - 2az + aa} + \frac{a^3z}{z - a} = z^3$. And by Reduction, $z^4 - 2az^3 + aazz - 2a^3z + a^4 = 0$.

In the like manner, having propos'd $\frac{xy}{c}=z$, and $cy+zx=cc$, to take away $z$, I ſubſtitute in its Room $\frac{xy}{c}$ in the ſecond Æquation, and there comes out $cy+\frac{xxy}{c}=cc$.

But a Perſon uſed to theſe Sorts of Computations, will oftentimes find ſhorter Methods [than theſe] by which the unknown Quantity may be exterminated. Thus, having $ax=\frac{bbx-b^3}{z}$, and $x=\frac{az}{x-b}$, if equal Quantities are multiply'd by Equals, there will come out equal Quantities, *viz.* $axx=abb$, or $x=b$.

But I leave particular Caſes of this Kind to be found out by the Students as Occaſion ſhall offer.

## *The Extermination of an unknown Quantity of ſeveral Dimenſions in each Æquation.*

WHEN the Quantity to be [exterminated or] taken away is of more than one Dimenſion in both the Æquations, the Value of its greateſt Power muſt be ſought in both; then if thoſe Powers are not the ſame, the Æquation that involves the leſſer Power muſt be multiply'd by the Quantity to be taken away, or by its Square, or Cube, &c. that it may become of the ſame Power with the other Æquation. Then the Values of thoſe Powers are to be made Equal, and there will come out a new Æquation, where the greateſt Power or Dimenſion of the Quantity to be taken away is diminiſh'd. And by repeating this Operation, the Quantity will at length be taken away.

As if you have $xx+5x=3yy$, and $2xy-3xx=4$, to take away $x$, the firſt [Æquation] will give $xx=-5x+3yy$, and the ſecond $xx=\frac{2xy-4}{3}$. I put therefore $3yy-5x=\frac{2xy-4}{3}$, and ſo $x$ is reduc'd to only one Dimenſion, and ſo may be taken away by what I have before ſhewn, *viz.* by a due Reduction of the laſt Æquation there comes out $9yy-15x=2xy-4$, or $x=$

$x = \frac{9yy+4}{2y+15}$. I therefore ſubſtitute this Value for $x$ in one of the Æquations firſt propoſ'd, (as in $xx+5x=3yy$) and there ariſes $\frac{81y^4+72yy+16}{4yy+60y+225} + \frac{45yy+20}{2y+15} = 3yy$. To reduce which into Order, I multiply by $4yy+60yy+225$, and there comes out $81y^4+72yy+16+90y^3+40y+675yy+300 = 12y^4+180y^3+675yy$, or $69y^4-90y^3+72yy+40y+316=0$.

Moreover, if you have $y^3 = xyy+3x$, and $yy = xx-xy-3$; to take away $y$, I multiply the latter Æquation by $y$, and you have $y^3 = xxy-xyy-3y$, of as many Dimenſions as the former. Now, by making the Values of $y^3$ equal to one another, I have $xyy+3x = xxy-xyy-3y$; where $y$ is depreſs'd to two Dimenſions. By this therefore, and the moſt Simple one of the Æquations firſt propoſ'd $yy = xx-xy-3$, the Quantity $y$ may be wholly taken away by the ſame Method as in the former Example.

There are moreover other Methods by which this may be done, and that oftentimes more conciſely. As therefore, if $yy = \frac{2x^2y}{a} + xx$, and $yy = 2xy + \frac{x^4}{aa}$; that $y$ may be extirpated, extract the Root $y$ in each, as is ſhewn in the 7th Rule, and there will come out $y = \frac{xx}{a} + \sqrt{\frac{x^4}{aa}+xx}$, and $y = x + \sqrt{\frac{x^4}{aa}+xx}$. Now, by making theſe two Values of $y$ equal, you'll have $\frac{xx}{a} + \sqrt{\frac{x^4}{aa}+xx} = x + \sqrt{\frac{x^4}{aa}+xx}$, and by rejecting the equal Quantities $\sqrt{\frac{x^4}{aa}+xx}$, there will remain $\frac{xx}{a} = x$, or $xx = ax$, and $x = a$.

Moreover, to take $x$ out of the Æquations $x+y+\frac{yy}{x} = 20$, and $xx+yy+\frac{y^4}{xx} = 140$, take away $y$ from the firſt Æquation, and there remains $x+\frac{yy}{x}=20-y$, and ſquaring the Parts $xx+2yy+\frac{y^4}{xx}=400-40y+yy$, and taking away $yy$ on both Sides, there remains $xx+yy+\frac{yyyy}{xx}=400-40y$. Wherefore, ſince $400-40y$, and 140 are equal to the ſame Quantities, $400-40y$ will $=140$, or $y=6\frac{1}{2}$; and ſo you may contract the Matter in moſt other Æquations.

But when the Quantity to be exterminated is of ſeveral Dimenſions, ſometimes there is requir'd a very laborious Calculus to exterminate it out of the Æquations; but then the Labour will be much diminiſh'd by the following Examples made Uſe of as Rules.

## RULE I.

From $axx+bx+c=0$, and $fxx+gx+h=0$. $x$ being exterminated, there comes out

$$\overline{ah-bg-2cf}\times ah+\overline{bh-cg}\times bf+\overline{agg+cff}\times c=0.$$

## RULE II.

From $ax^3+bxx+cx+d=0$, and $fxx+gx+h=0$. $x$ being exterminated, there comes out

$$\overline{ah-bg-2cf}\times ahh+\overline{bh-cg-2df}\times bfh+\overline{ch-dg}\times\overline{agg+cff}+\overline{3agh+bgg+dff}\times df=0.$$

## RULE III.

From $ax^4+bx^3+cxx+dx+e=0$, and $fxx+gx+h=0$. $x$ being exterminated, there comes out

$$\overline{ah-bg-2cf}\times ah^3+\overline{bh-cg-2df}\times bfhh+\overline{agg+cff}\times\overline{chh-dgh+egg-2efh}+\overline{3agh+bgg+dff}\times dfh+\overline{2ahh+3bgh-dfg+eff}\times eff-\overline{bg-2ah}\times efgg=0.$$

## RULE IV.

From $ax^3 + bxx + cx + d = 0$, and $fx^3 + gx^2 + hx + k = 0$.

$x$ being exterminated, there comes out

$\overline{ah - bg - 2cf} \times \overline{adhh - achk} + \overline{ak + bh - cg - 2df} \times \overline{bdfh - ak + bh + 2cg + 3df} \times aakk : + \overline{cdh - ddg - cck + 2bdk} \times \overline{agg + cff} : + \overline{3agh + bgg + dff - 3afk} \times dd \overline{- 3ak - bh + cg + df} \times bcfk + \overline{bk - 2dg} \times bbfk : \overline{- bbk - 3adh - cdf} \times agk = 0$.

For Example, to exterminate $x$ out of the Æquations $xx + 5x - 3yy = 0$, and $3xx - 2xy + 4 = 0$: I reſpectively ſubſtitute in the firſt Rule for $a, b, c; f, g,$ and $h$ [theſe Quantities, *viz.*] $1, 5, -3yy; 3, -2y$ and $4$; and duly obſerving the Signs + and —, there ariſes $\overline{4 + 10y + 18yy} \times 4 + \overline{20 - by^3} \times 15 + \overline{4yy - 27yy} \times -3yy = 0$, or $16 + 40y + 72yy + 300 - 90y^3 + 69y^4 = 0$.

By the like Reaſon that $y$ may be expung'd out of the Æquations $y^3 - xyy - 3x = 0$, and $yy + xy - xx + 3 = 0$, I ſubſtitute into the ſecond Rule for $a, b, c, d; f, g, h,$ and $x$, [theſe Quantities] $1, -x, 0, -3x; 1, x, \overline{-xx + 3}$, and $y$ reſpectively, and there comes out $\overline{3 - xx + xx} \times \overline{9 - 6xx + x^4} - \overline{3x + x^3} + \overline{6x} \times -3x + x^3 : + 3xx \times xx + 9x - 3x^3 - x^3 - 3x \times -3x = 0$. Then blotting out the ſuperfluous Quantities and multiplying, you have $27 - 18xx + 3x^4, - 9xx + x^6, + 3x^4 - 18x^2 + 12x^4 = 0$. And ordering (duely) $x^6 + 18x^4 - 45xx + 27 = 0$.

Hitherto [we have diſcours'd] of taking away one unknown Quantity out of two Æquations. Now, if ſeveral are to be taken out of ſeveral, the Buſineſs muſt be done by degrees: Out of the Æquations $ax = yz$, $x + y = 2$, and $5x = y + 3z$; if the Quantity $y$ is to be found, firſt, take out one of the Quantities $x$ or $z$, ſuppoſe $x$, by ſubſtituting for it, its Value $\frac{yz}{a}$ (found by the firſt Æquation) in the ſecond and third Æquations; and then you will have $\frac{z}{a} + y = z$, and $\frac{5yz}{a} = y + 3z$, out of which take away $z$ as above.

## *Of the Method of taking away any Number of Surd Quantities out of Æquations.*

Hitherto may be referred the Extermination of Surd Quantities, by making them equal to any [other] Letters. As if you have $\sqrt{ay} - \sqrt{aa - ay} = 2a + \sqrt[3]{ayy}$, by writing $t$ for $\sqrt{ay}$, and $v$ for $\sqrt{aa - ay}$, and $x$ for the $\sqrt[3]{ayy}$. you'll have the Æquations $t - v = 2a + x$, $tt = ay$, $vv = aa - ay$, and $x^3 = ayy$, out of which taking away by degrees $t$, $v$, and $x$, there will reſult an Æquation entirely free from Surdity.

## *How a Queſtion may be brought to an Æquation.*

AFTER the Learner has been ſome Time exerciſed in managing and transforming Æquations, Order requires that he ſhould try his Skill in bringing Queſtions to an Æquation. And any Queſtion being propoſed, his Skill is particularly required to denote all its Conditions by ſo many Æquations. To do which, he muſt firſt conſider whether the Propoſitions or Sentences in which it is expreſs'd, be all of them fit to be denoted in Algebraick Terms, juſt as we expreſs our Conceptions in *Latin* or *Greek* Characters. And if ſo, (as will happen in Queſtions converſant about Numbers or abſtract Quantities) then let him give Names to both known and unknown Quantities, as far as Occaſion requires. And the Conditions thus tranſlated to Algebraick Terms will give as many Æquations as are neceſſary to ſolve it.

As if there are required three Numbers in continual Proportion whoſe Sum is 20, and the Sum of their Squares 140; putting $x$, $y$, and $z$ for the Names of the three Numbers ſought, the Queſtion will be tranſlated out of the Verbal to the Symbolical Expreſſion, as follows:

| *The Queſtion in Words.* | *The ſame in Symbols.* |
|---|---|
| There are ſought three Numbers on theſe Conditions: | $x, y, z$? |
| That they ſhall be continually proportional. | $x:y::y:z$, or $xz=yy$. |
| That the Sum ſhall be 20. | $x+y+z=20$. |
| And the Sum of their Squares 140. | $xx+yy+zz=140$. |

And ſo the Queſtion is brought to [theſe] Æquations, *viz.* $xz=yy$, $x+z+y=20$, and $xx+yy+zz=140$, by the Help whereof $x$, $y$, and $z$, are to be found by the Rules deliver'd above.

But you muſt note, That the Solutions of Queſtions are (for the moſt part) ſo much the more expedite and artificial, by how fewer unknown Quantities you have at firſt. Thus, in the Queſtion propos'd, putting $x$ for the firſt Number, and $y$ for the ſecond, $\frac{yy}{x}$ will be the third Proportional; which then being put for the third Number, I bring the Queſtion into Æquations, as follows:

| *The Queſtion in Words.* | *Symbolically.* |
|---|---|
| There are ſought three Numbers in continual Proportion. | $(x, y, \frac{yy}{x}$?) |
| Whoſe Sum is 20. | $x+y+\frac{yy}{x}=20$. |
| And the Sum of their Squares 140. | $xx+yy+\frac{y^4}{xx}=140$. |

You have therefore the Æquations $x+y+\frac{yy}{x}=20$, and $xx+yy+\frac{y^4}{xx}=140$, by the Reduction whereof $x$ and $y$ are to be determined.

Take another Example. A certain Merchant encreaſes his Eſtate yearly by a third Part, abating 100 *l.* which he ſpends yearly in his Family; and after three Years he finds his Eſtate doubled. *Query*, What he is worth?

To reſolve this, you muſt know there are [or lie hid] ſeveral Propoſitions, which are all thus found out and laid down.

| *In Engliſh.* | *Algebraically.* |
|---|---|
| A Merchant has an Eſtate—— —— | $x$. |
| Out of which the firſt Year he expends 100 *l.* | $x - 100$. |
| And augments the reſt by one third. —— | $x - 100 + \frac{x - 100}{3}$, or $\frac{4x - 400}{3}$. |
| And the ſecond Year expends 100 *l.* | $\frac{4x - 400}{3} - 100$, or $\frac{4x - 700}{3}$. |
| And augments the reſt by a third —— —— | $\frac{4x - 700}{3} - \frac{4x - 700}{9}$, or $\frac{16x - 2800}{9}$. |
| And ſo the third Year expends 100 *l.* —— | $\frac{16x - 2800}{9} - 100$, or $\frac{16x - 3700}{9}$. |
| And by the reſt gains likewiſe one third Part | $\frac{16x - 3700}{9} + \frac{16x - 3700}{27}$, or $\frac{64x - 14800}{27}$. |
| And he becomes [at length] twice as rich as at firſt—— —— | $\frac{64x - 14800}{27} = 2x$. |

Therefore the Queſtion is brought to this Æquation; $\frac{64x - 14800}{27} = 2x$, by the Reduction whereof you are to find $x$; *viz.* Multiply it by 27, and you have $64x = 14800 = 54x$; ſubtract $54x$, and there remains $10x - 14800 = 0$, or $10x = 14800$, and dividing by 10, you have $x = 1480$. Wherefore, 1480 *l.* was his Eſtate at firſt, as alſo his Profit or Gain ſince.

You ſee therefore, that to the Solution of Queſtions which only regard Numbers, or the abſtracted Relations of Quantities, there is ſcarce any Thing elſe required than that the Problem be tranſlated out of the *Engliſh*, or any other Tongue it is propos'd in, into the Algebraical Language, that is, into Characters fit to denote our Conceptions of the Relations of Quantities. But it may ſometimes happen, that the Language [or the Words] wherein the State of the Queſtion is expreſs'd, may ſeem unfit to be turn'd into the Algebraical Language; but making Uſe of a few Changes, and attending to the Senſe rather than the Sound of the Words, the Verſion will become eaſy. Thus, the Forms of Speech among [ſeveral] Nations have their proper Idioms; which, where they happen, the Tranſlation out of one into another is not to be made literally, but to be determin'd by the Senſe. But that I may illuſtrate theſe Sorts of Problems, and make familiar the Method of reducing them to Æquations; and ſince Arts are more eaſily learn'd by Examples than Precepts, I have thought fit to adjoin the Solutions of the following Problems.

PROBLEM I. Having given the Sum of two Numbers ($a$), and the Difference of their Squares ($b$), to find the Numbers?

Let the leaſt of them be [call'd] $x$, the other will be $a-x$, and their Squares $xx$, and $aa-2ax+xx$ the Difference, whereof $aa-2ax$ is ſuppos'd $b$. Therefore, $aa-2ax=b$, and then by Reduction $aa-b=2ax$, or $\frac{aa-b}{2a}\left(=\frac{1}{2}a-\frac{b}{2a}\right)=x$. For Example, if the Sum of the Numbers, or $a$, be 8, and the Difference of the Squares, or $b$, be 16; $\frac{1}{2}a-\frac{b}{2a}$ will be $(=4-1)=3=x$, and $a-x=5$. Wherefore the Numbers are 3 and 5.

PROBLEM II. To find three Quantities, $x$, $y$, and $z$, the Sum of any two of which ſhall be given.

If the Sum of two of them, *viz.* $x$ and $y$, be $a$; of $x$ and $z$, $b$; and of $y$ and $z$, $c$; there will be had three Æquations to determine the three Quantities ſought, $x$, $y$, and $z$, *viz.* $x+y=a$, $x+z=b$, and $y+z=c$. Now, that two of the unknown Quantities, *viz.* $y$ and $z$ may be exterminated, take away $x$ on both Sides in the firſt and ſecond Æquation, and you'll have $y=a-x$, and $z=b-x$, which Values ſubſtitute for $y$ and $z$ in the third [Æquation], and there will come out $a-x+b-x=c$, and by Reduction $x=\frac{a+b-c}{2}$; and having found $x$, the Æquations above $y=a-x$, and $z=b-x$, will give $y$ and $z$.

Example. If the Sum of $x$ and $y$ be 9, of $x$ and $z$, 10, and $y$ and $z$, 13; then, in the Values of $x$, $y$, and $z$, write 9 for $a$, 10 for $b$, and 13 for $c$, and you'll have $a + b - c = 6$, and conſequently $x \left(= \frac{a+b-c}{2}\right) = 3$, $y (= a - x) = 6$, and $z (= b - x) = 7$.

Problem III. To divide a given Quantity into as many Parts as you pleaſe, ſo that the greater Parts may exceed the leaſt by [any] given Differences.

Let ($a$) be a Quantity to be divided into four ſuch Parts, and its firſt or leaſt Part call $x$, and the Exceſs of the ſecond Part above this call $b$, and of the third Part $c$, and of the fourth $d$; and $x + b$ will be the ſecond Part, $x + c$ the third, and $x + d$ the fourth, the Aggregate of all which $4x + b + c + d$ is equal to the whole Line $a$. Take away on both Sides $b + c + d$, and there remains $4x = a - b - c - d$, or $x = \frac{a-b-c-d}{4}$.

Example. Let there be propoſed a Line of 20 Foot, ſo to be divided into four Parts, that the Exceſs of the ſecond above the firſt Part ſhall be 2 Foot, of the third 3 Foot, and of the fourth ſeven Foot, and the four Parts will be $x \left(= \frac{a-b-c-d}{4}, \text{ or } \frac{20-2-3-7}{4}\right) = 2$, $x + b = 4$. $x + c = 5$, and $x + d = 9$. After the ſame Manner a Quantity is divided into more Parts on the ſame Conditions.

Problem IV. A Perſon being willing to diſtribute ſome Money among ſome Beggars, wanted eight Pence to give three Pence a peice to them; he therefore gave to each two Pence, and had three Pence remaining over and above. To find the Number of the Beggars.

Let the Number of the Beggars be $x$, and there will be wanting eight Pence to give all $3x$ [Number of] Pence, he has therefore $3x - 8$ Pence; out of theſe he gives $2x$ Pence, and the remaining Pence $x - 8$ are three. That is, $x - 8 = 3$, or $x = 11$.

PROBLEM V. If two Poſt-Boys, *A* and *B*, at 59 Miles Diſtance from one another, meet in the Morning, of whom *A* rides 7 Miles in two Hours, and *B* 8 Miles in three Hours, and *B* ſets out one Hour later than *A*; to find what Number of Miles *A* will ride before he meets *B*.

Call that Length $x$, and you'll have $59 - x$, the Length of *B*'s Journey. And ſince *A* travels 7 Miles in two Hours, he will make the Space $x$ in $\frac{2x}{7}$ Hours, becauſe 7 Miles : 2 Hours : : $x$ Miles : $\frac{2x}{7}$ Hours. And ſo, ſince *B* rides 8 Miles in 3 Hours, he will deſcribe his Space [or ride his Journey] $59 - x$ in $\frac{177-3x}{8}$ Hours. Now, ſince the Difference of theſe Times is one Hour, to the End they may become equal, add that Difference to the ſhorter Time $\frac{177-3x}{8}$, and you'll have $1 + \frac{177-3x}{8} = \frac{2x}{7}$, and by Reduction $35 - x$. For, multiplying by 8 you have $185 - 3x = \frac{16x}{7}$. Then multiplying alſo by 7 you have $1295 - 21x = 16x$, or $1295 = 37x$. And, laſtly, dividing by 37, there ariſes $35 = x$. Therefore, 35 Miles is the Diſtance that *A* muſt ride before he meets *B*.

### *The ſame more generally.*

Having given the [Velocities] Celerities [or Swiftneſſes] of two moveable Bodies, *A* and *B*, tending to the ſame Place, together with the Interval [or Diſtance] of the Places and Times from and in which they begin to move; to determine the Place they ſhall meet in.

Suppoſe, the Velocity of the Body *A* to be ſuch, that it ſhall paſs over the Space $c$ in the Time $f$; and of the Body *B* to be ſuch as ſhall paſs over the Space $d$ in the Time $g$; and that the Interval of the Places is $e$, and $h$ the Interval of the Times in which they begin to move.

CASE I. Then if both tend to the ſame Place, [or the ſame Way] and *A* be the Body that, at the Beginning of the Motion, is fartheſt diſtant from the Place they tend to:

call that Diſtance $x$, and ſubtract from it the Diſtance $e$, and there will remain $x-e$ for the Diſtance of *B* from the Place it tends to. And ſince *A* paſſes through the Space $c$ in the Time $f$, the Time in which it will paſs over the Space $x$ will be $\frac{fx}{c}$, becauſe the Space $c$ is to the Time $f$, as the Space $x$ to the Time $\frac{fx}{c}$. And ſo, ſince *B* paſſes the Space $d$ in [the Time] $g$, the Time in which it will paſs the Space $x-e$ will be $\frac{gx-ge}{d}$. Now ſince the Difference of theſe Times is ſuppoſed $h$, that they may become equal, add $h$ to the ſhorter Time, *viz.* to the Time $\frac{fx}{c}$ if *B* begins to move firſt, and you'll have $\frac{fx}{c}+h=\frac{gx-ge}{d}$, and by Reduction $\frac{cge+cdh}{cg-df}$, or $\frac{ge+dh}{g-\frac{df}{c}}=x$. But if *A* begins to move firſt, add $h$ to the Time $\frac{gx-ge}{d}$, and you'll have $\frac{fx}{c}=h+\frac{gx-ge}{d}$, and by Reduction $\frac{cge-cdh}{cg-df}=x$.

Case II. If the moveable Bodies meet, and $x$, as before, be made the initial Diſtance of the moveable Body *A*, from the Place it is to move to, then $c-x$ will be the initial Diſtance of the Body *B* from the ſame Place; and $\frac{fx}{c}$ the Time in which *A* will deſcribe the Diſtance $x$, and $\frac{ge-gx}{d}$ the Time in which *B* will deſcribe its Diſtance $e-x$. To the leſſer of which Times, as above, add the Difference $h$, *viz.* to the Time $\frac{fx}{c}$ if *B* begin firſt to move, and ſo you'll have $\frac{fx}{c}+h=\frac{ge-gx}{d}$, and by Reduction $\frac{cge-cdh}{cg+df}=x$.

**Example I.** If the Sun moves every Day one Degree, and the Moon thirteen, and at a certain Time the Sun be at the Beginning of *Cancer*, and, in three Days after, the Moon in the Beginning of *Aries*, the Place of their next following Conjunction is demanded. Anſwer in $10\frac{3}{4}$ Deg. of *Cancer*. For ſince they both are going towards the ſame Parts, and the Motion of the Moon, which is farther diſtant from the Conjunction, hath a later *Epocha*, the Moon will be *A*, the Sun *B*, and $\frac{cge+cdh}{cg-df}$ the Length of the Moon's Way, which, if you write 13 for $c$, 1 for $f$, $d$, and $g$, 90 for $e$, and 3 for $h$, will become $\frac{13\times1\times90+13\times1\times3}{13\times1-1\times1}$, that is, $\frac{1209}{12}$, or $100\frac{3}{4}$ Degrees; and then add theſe Degrees to the Beginning of *Aries*, and there will come out $10\frac{3}{4}$ Deg. of *Cancer*.

**Example II.** If two Poſt-Boys, *A* and *B*, being in the Morning 59 Miles aſunder, ſet out to meet each other, and *A* goes 7 Miles in 2 Hours, and *B* 8 Miles in 3 Hours, and *B* begins his Journey 1 Hour later than *A*, it is demanded how far *A* will have gone before he meets *B*. Anſwer, 35 Miles. For ſince they go towards each other, and *A* ſets out firſt, $\frac{cge+cdh}{cg+df}$ will be the Length of his Journey; and writing 7 for $c$, 2 for $f$, 8 for $d$, 3 for $g$, 59 for $e$, and 1 for $h$, this will become $\frac{7\times3\times59+7\times8\times1}{7\times3+8\times2}$, that is, $\frac{1295}{37}$, or 35,

**Problem VI.** Giving the Power of any Agent, to find how many ſuch Agents will perform a given Effect $a$ in a given Time $b$.

Let the Power of the Agent be ſuch that it can produce the Effect $c$ in the Time $d$, and it will be as the Time $d$ to the Time $b$, ſo the Effect $c$ which that Agent can produce in the Time $d$ to the Effect which he can produce in the Time $b$, which then will be $\frac{bc}{d}$. Again, as the Effect of one Agent $\frac{bc}{d}$ to the Effect of all $a$; ſo that ſingle Agent to all the Agents; and thus the Number of the Agents will be $\frac{ad}{bc}$.

Example. If a Scribe can in 8 Days write 15 Sheets, how many ſuch Scribes muſt there be to write 405 Sheets in 9 Days? Anſwer 24. For if 8 be ſubſtituted for $d$, 15 for $c$, 405 for $a$, and 9 for $b$, the Number $\frac{ad}{bc}$ will become $\frac{405\times8}{9\times15}$, that is, $\frac{3240}{135}$, or 24.

Problem VII. The Forces of ſeveral Agents being given, to determine $x$ the Time, wherein they will joyntly perform a given Effect $d$.

Let the Forces of the Agents $A$, $B$, $C$ be ſuppoſed, which in the Times $e$, $f$, $g$ can produce the Effects $a$, $b$, $c$ reſpectively; and theſe in the Time $x$ will produce the Effects $\frac{ax}{e}$, $\frac{bx}{f}$, $\frac{cx}{g}$; wherefore is $\frac{ax}{e}+\frac{bx}{f}+\frac{cx}{g}=d$, and by Reduction $x=\frac{d}{\frac{a}{e}+\frac{b}{f}+\frac{c}{g}}$.

Example. Three Workmen can do a Piece of Work in certain Times, *viz.* $A$ once in 3 Weeks, $B$ thrice in 8 Weeks, and $C$ five times in 12 Weeks. It is deſired to know in what Time they can finiſh it joyntly? Here then are the Forces of the Agents $A$, $B$, $C$, which in the Times 3, 8, 12 can produce the Effects 1, 3, 5 reſpectively, and the Time is ſought wherein they can do one Effect. Wherefore, for $a$, $b$, $c$, $d$, $e$, $f$, $g$ write 1, 3, 5, 1, 3, 8, 12, and there will ariſe $x=\frac{1}{\frac{1}{3}+\frac{3}{8}+\frac{5}{12}}$, or $\frac{8}{9}$ of a Week, that is, [allowing 6 working Days to a Week, and 12 Hours to each Day] 5 Days and 4 Hours, the Time wherein they will joyntly finiſh it.

Problem VIII. So, to compound unlike Mixtures of two or more Things, that the Things mix'd together may have a given Ratio to one another, Let the given Quantity of one Mixture be $dA+eB+fC$, the ſame Quantity of another Mixture $gA+hB+kC$, and the ſame of a third $lA+mB+nC$, where $A, B, C$ denote the Things mix'd, and $d, e, f, g, h,$ &c. the Proportions of the ſame in the Mixtures. And let $pA+qB+rC$ be the Mixture which muſt be compos'd of the three Mixtures; and ſuppoſe $x, y,$ and $z$ to be the Numbers, by which if the three given Mixtures be reſpectively multiply'd, their Sum will become $pA+qB+rC$.

$$\text{Therefore is}\left\{\begin{array}{l} dxA+exB+fxC\\ +gyA+hyB+kyC\\ +lzA+mzB+nzC\end{array}\right\}=pA+qB+rC.$$

And then making $dx+gy+lz=p$; $ex+hy+mz=q$, and $fx+ky+nz=r$, and by Reduction $x=\frac{p-gy-lz}{d}=\frac{r-ky-nz}{f}$. And again, the Æquations $\frac{p-gy-lz}{d}=\frac{q-hy-mz}{e}$, and $\frac{q-hy-mz}{e}=\frac{r-ky-nz}{f}$ by Reduction give $\frac{ep-dq+dmz-elz}{eg-dh}\;(=y)=\frac{fq-er+enz-fmz}{fh-ek}$, which, if abbreviated by writing $\alpha$ for $ep-dq$, $\beta$ for $dm-el$, $\gamma$ for $eg-dh$, $\delta$ for $fq-er$, $\zeta$ for $en-fm$, and $\theta$ for $fh-ek$, will become $\frac{\alpha+\beta z}{\gamma}=\frac{\delta+\zeta z}{\theta}$, and by Reduction $\frac{\theta\alpha-\gamma\delta}{\gamma\zeta-\beta\theta}=z$. Having found $z$, put $\frac{\alpha+\beta z}{\gamma}=y$, and $\frac{p-gy-lz}{d}=x$.

EXAMPLE. If there were three Mixtures of Metals melted down together; of the firſt of which a Pound [Averdupois] contains of Silver ℥ 12, of Braſs ℥ 1, and of Tin ℥ 3; of the ſecond, a Pound contains of Silver ℥ 1, of Braſs ℥ 12, and of Tin ℥ 3; and a Pound of the third contains of Braſs ℥ 14, of Tin ℥ 2, and no Silver; and let theſe Mixtures be ſo to be compounded, that a Pound of the Compoſition may contain of Silver ℥ 4, of Braſs ℥ 9, and of Tin ℥ 3: For $d, e, f;\ g, h, k;\ l, m, n;\ p, q, r,$ write 12, 1, 3; 1, 12, 3; 0, 14, 2; 4, 9, 3 reſpectively, and $\alpha$ will be $(=ep-dq=1\times4-12\times9)=-104$, and $\beta$ $(=dm-el=12\times14-1\times0)=168$, and ſo $\gamma=-143$, $\delta=24$, $\zeta=-40$,

$= -40$, and $\theta = 33$. And therefore $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-3432 + 3432}{5720 - 5544}\right) = 0$; $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-104 + 0}{-143}\right) = \frac{8}{11}$, and $x \left(= \frac{p - gy - lz}{d} = \frac{4 - \frac{8}{11}}{12}\right) = \frac{3}{11}$. Wherefore, if there be mix'd $\frac{8}{11}$ Parts of a Pound of the ſecond Mixture, $\frac{3}{11}$ Parts of a Pound of the third, and nothing of the firſt, the Aggregate will be a Pound, containing four Ounces of Silver, nine of Braſs, and three of Tin.

PROBLEM IX. The Prices of ſeveral Mixtures of the ſame Things, and the Proportions of the Things mix'd together being given, to determine the Price of each of the Things mix'd.

Of each of the Things $A$, $B$, $C$, let the Price of the Mixture $dA + gB + lC$ be $p$, of the Mixture $eA + hB + mC$ the Price $q$, and of the Mixture $fA + kB + nC$ the Price $r$, and of thoſe Things $A$, $B$, $C$ let the Prices $x$, $y$, $z$ be demanded. For the Things $A$, $B$, $C$ ſubſtitute their Prices $x$, $y$, $z$, and there will ariſe the Æquations $dx + gy + lz = p$, $ex + hy + mz = q$, and $fx + ky + nz = r$; from which, by proceeding as in the foregoing Problem, there will in like manner be got $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = z$, $\frac{\alpha + \beta z}{\gamma} = y$, and $\frac{p - gy - lz}{d} = x$.

EXAMPLE. One bought 40 Buſhels of Wheat, 24 Buſhels of Barley, and 20 Buſhels of Oats together, for 15 Pounds 12 Shillings. Again, he bought of the ſame Grain 26 Buſhels of Wheat, 30 Buſhels of Barley, and 50 Buſhels of Oats together, for 16 Pounds. And thirdly, he bought of the like kind of Grain, 24 Buſhels of Wheat, 120 Buſhels of Barley, and 100 Buſhels of Oats together, for 34 Pounds. It is demanded at what Rate a Buſhel of each of the Grains ought to be valued. Anſwer, a Buſhel of Wheat at 5 Shillings, of Barley at 3 Shillings, and of Oats at 2 Shillings. For inſtead of $d, g, l$; $e, h, m$; $f, k, n$; $p, q, r$, by writing reſpectively 40, 24, 20; 26, 30, 50; 24, 120, 100; $15\frac{3}{5}$, 16, and 34, there ariſes $\alpha$ $(= ep - dq = 26 \times 15\frac{3}{5} - 40 \times 16) = -234\frac{2}{5}$, and $\beta$ $(= dm - el = 40 \times 50 - 26 \times 20) = 1480$, and thus $\gamma = -576$, $\delta = -500$, $\zeta = 1400$, and $\theta = -2400$. Then $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{562560 - 288000}{-806400 + 3552000} = \frac{274560}{2745600}\right) = \frac{1}{10}$; $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-234\frac{2}{5} + 148}{-576}\right) = \frac{3}{20}$; and $x \left(= \frac{p - gy - lz}{d} = \frac{15\frac{3}{5} - \frac{18}{5} - 2}{40}\right) = \frac{1}{4}$. Therefore a Buſhel of Wheat coſt $\frac{1}{4}$ ℔, or 5 Shillings, a Buſhel of Barley $\frac{3}{20}$ ℔, or 3 Shillings, and a Buſhel of Oats $\frac{1}{10}$ ℔, or 2 Shillings.

**Problem X.** There being given the ſpecifick Gravity both of the Mixture and the [two] Things mix'd, to find the Proportion of the mix'd Things to one another.

Let $e$ be the ſpecifick Gravity of the Mixture $\overline{A+B}$, $a$ the ſpecifick Gravity of $A$, and $b$ the ſpecifick Gravity of $B$; and ſince the abſolute Gravity, or the Weight, is compoſed of the Bulk of the Body and the ſpecifick Gravity, $aA$ will be the Weight of $A$, $bB$ of $B$, and $eA + eB$ the Weight of the Mixture $\overline{A+B}$; and therefore $aA + bB = eA + eB$; and from thence $aA - eA = eB - bB$; and conſequently $e - b : a - e :: A : B$.

**Example.** Suppoſe the Gravity [or ſpecifick Weight] of Gold to be as 19, and of Silver as $10\frac{1}{3}$, and [King] *Hiero*'s Crown as 17; and $[6\frac{2}{3} : 2] :: 10 : 3$ $(e - b : a - e :: A : B)$ :: Bulk of Gold in the Crown : Bulk of Silver, or $190 : 31$ $(:: 19 \times 10 : 10\frac{1}{3} \times 3 :: a \times \overline{e - b} : b \times \overline{a - e})$ :: as the Weight of Gold in the Crown, to the Weight of Silver, and $221 : 31$ :: as the Weight of the Crown to the Weight of the Silver.

**Problem XI.** If the Number of Oxen $a$ eat up the Meadow $b$ in the Time $c$; and the Number of Oxen $d$ eat up as good a Piece of Paſture $e$ in the Time $f$, and the Graſs grows uniformly; to find how many Oxen will eat up the like Paſture $g$ in the Time $h$.

If the Oxen $a$ in the Time $c$ eat up the Paſture $b$; then by Proportion, the Oxen $\frac{e}{b}a$ in the ſame Time $c$, or the Oxen $\frac{ec}{bf}a$ in the Time $f$, or the Oxen $\frac{ec}{bh}a$ in the Time $h$

will eat up the Paſture $e$; ſuppoſing the Graſs did not grow [at all] after the Time $c$. But ſince, by reaſon of the Growth of the Graſs, all the Oxen $d$ in the Time $f$ can eat up only the Meadow $e$, therefore that Growth of the Graſs in the Meadow $e$, in the Time $f-c$, will be ſo much as alone would be ſufficient to feed the Oxen $d-\frac{eca}{bf}$ the Time $f$, that is as much as would ſuffice to feed the Oxen $\frac{df}{h}-\frac{eca}{bh}$ in the Time $h$. And in the Time $h-c$, by Proportion, ſo much would be the Growth of the Graſs as would be ſufficient to feed the Oxen $\frac{h-c}{f-c}$ into $\overline{\frac{df}{h}-\frac{eca}{bh}}$ or $\frac{bdfh-ecah-bdcf+aecc}{bfh-bch}$. Add this Increment to the Oxen $\frac{aec}{bh}$, and there will come out $\frac{bdfh-ecah-bdcf+ecfa}{bfh-bch}$, the Number of Oxen which the Paſture $e$ will ſuffice to feed in the Time $h$. And ſo by [in] Proportion the Meadow $g$ will ſuffice to feed the Oxen $\frac{gbdfh-ecagh-bdcgf+ecfga}{befh-bceh}$ during the ſame Time $h$.

**Example.** If 12 Oxen eat up $3\frac{1}{3}$ Acres of Paſture in 4 Weeks, and 21 Oxen eat up 10 Acres of like Paſture in 9 Weeks; to find how many Oxen will eat up 36 Acres in 18 Weeks? Anſwer 36; for that Number will be found by ſubſtituting in $\frac{bdfgh-ecagh-bdcgf+ecfga}{befh-bceh}$ the Numbers 12, $3\frac{1}{3}$, 4, 21, 10, 9, 36, and 18 for the Letters $a$, $b$, $c$, $d$, $e$, $f$, $g$, and $h$ reſpectively; but the Solution, perhaps, will be no leſs expedite, if it be brought out from the firſt Principles, in Form of the precedent literal Solution. As if 12 Oxen in 4 Weeks eat up $3\frac{1}{3}$ Acres, then by Proportion 36 Oxen in 4 Weeks, or 16 Oxen in 9 Weeks, or 8 Oxen in 18 Weeks, will eat up 10 Acres, on Suppoſition that the Graſs did not grow. But ſince by reaſon of the Growth of the Graſs 21 Oxen in 9 Weeks can eat up only 10 Acres, that Growth of the Graſs in 10 Acres for the laſt 5 Weeks will be as much as would be ſufficient to feed 5

Oxen, that is the Exceſs of 21 above 16 for 9 Weeks, or, what is the ſame Thing, to feed $\frac{5}{2}$ Oxen for 18 Weeks. And in 14 Weeks (the Exceſs of 18 above the firſt 4) the Increaſe of the Graſs, by Analogy, will be ſuch, as to be ſufficient to feed 7 Oxen for 18 Weeks: Add theſe 7 Oxen, which the Growth of the Graſs alone would ſuffice to feed, to the 8, which the Graſs without Growth after 4 Weeks would feed, and the Sum will be 15 Oxen. And, laſtly, if 10 Acres ſuffice to feed 15 Oxen for 18 Weeks, then, in Proportion, 24 Acres would ſuffice 36 Oxen for the ſame Time.

Problem XII. Having given the Magnitudes and Motions of Spherical Bodies perfectly elaſtick, moving in the ſame right Line, and meeting one another, to determine their Motions after Reflexion.

The Reſolution of this Queſtion depends on theſe Conditions, that each Body will ſuffer as much by Re-action as the Action of each is upon the other, and that they muſt recede from each other after Reflexion with the ſame Velocity or Swiftneſs as they met before it. Theſe Things being ſuppos'd, let the Velocity of the Bodies $A$ and $B$, be $a$ and $b$ reſpectively; and their Motions (as being compos'd of their Bulk and Velocity together) will be $aA$ and $bB$. And if the Bodies tend the ſame Way, and $A$ moving more ſwiftly follows $B$, make $x$ the Decrement of the Motion $aA$, and the Increment of the Motion $bB$ ariſing by the Percuſſion; and the Motions after Reflexion will be $aA - x$ and $bB + x$; and the Celerities $\frac{aA - x}{A}$ and $\frac{bB + x}{B}$, whoſe Difference is $=$ to $a - b$ the Difference of the Celerities before Reflexion. Therefore there ariſes this Æquation $\frac{bB + x}{B} - \frac{aA + x}{A} = a - b$, and thence by Reduction $x$ becomes $= \frac{2aAB - 2bAB}{A + B}$, which being ſubſtituted for $x$ in the Celerities $\frac{aA - x}{A}$, and $\frac{bB + x}{B}$, there comes out $\frac{aA - aB + 2bB}{A + B}$ for the Celerity of $A$, and $\frac{2aA - bA + bB}{A + B}$ for the Celerity of $B$ after Reflexion.

But if the Bodies meet, then changing the Sign of *b*, the Velocities after Reflexion will be $\frac{aA-aB-2bB}{A+B}$, and $\frac{2aA+bA-bB}{A+b}$; either of which, if they come out, by Chance, Negative, it argues that Motion, after Reflexion, to tend a contrary Way to that which *A* tended to before Reflexion. Which is alſo to be underſtood of *A*'s Motion in the former Caſe.

Example. If the homogeneous Bodies [or Bodies of the ſame Sort] *A* of 3 Pound with 8 Degrees of Velocity, and *B* a Body of 9 Pounds with 2 Degrees of Velocity, tend the ſame Way; then for *A*, *a*, *B*, and *b*, write 3, 8, 9, and 2; and $\left(\frac{aA-aB+2bB}{A+B}\right)$ becomes $-1$, and $\left(\frac{2aA-bA+bB}{A+B}\right)$ becomes 5. Therefore *A* will return back with one Degree of Velocity after Reflexion, and *B* will go on with 5 Degrees.

Problem XIII. To find 3 Numbers in continual Proportion, whoſe Sum ſhall be 20, and the Sum of their Squares 140?

Make the firſt of the Numbers $=x$, and the ſecond $=y$, and the third will be $\frac{yy}{x}$, and conſequently $x+y+\frac{yy}{x}=20$; and $xx+yy+\frac{y^4}{xx}=140$. And by Reduction $xx\begin{array}{l}+y\\-20\end{array}x+yy=0$, and $x^4\begin{array}{l}+yy\\-140\end{array}xx+y^4=0$. Now to exterminate *x*, for *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, in the third Rule, ſubſtitute reſpectively $1, 0, yy-140, 0, y^4$; $1, y-20$, and $yy$: and there will come out $\overline{-yy+280}\times y^6 : +\overline{2yy-40y}\times\overline{260y^4-40y^5} : +3y^4\times y^4 : -2yy\times y^6-40y^5+400y^4 := 0$; and by Multiplication $1600y^6-10400y^5=0$, or $y=6\frac{1}{2}$. Which is found more ſhort by another Method before, but not ſo obvious as this. Moreover, to find *x*, ſubſtitute $6\frac{1}{2}$ for *y* in the Æquation $xx\begin{array}{l}+y\\-20\end{array}x+yy=0$, and there will ariſe $xx-13\frac{1}{2}x$

$+42\frac{1}{4}=0$, or $xx=13\frac{1}{2}x-42\frac{1}{4}$, and having extracted the Root $x=6\frac{3}{4}+$, or $-\sqrt{3\frac{5}{16}}$; *viz.* $6\frac{3}{4}+\sqrt{3\frac{5}{16}}$ is the greateſt of the three Numbers ſought, and $6\frac{3}{4}$ and $\sqrt{3\frac{5}{16}}$ the leaſt. For $x$ denotes ambiguouſly either of the extreme Numbers, and thence there will come out two Values, either of which may be $x$, the other being $\frac{yy}{x}$.

The ſame otherwiſe. Putting the Numbers $x$, $y$, and $\frac{yy}{x}$ as before, you'll have $x+y+\frac{yy}{x}=20$, or $xx=\left.\begin{array}{c}20\\ \overline{-y}\end{array}\right.x-yy$, and extracting the Root $x=10-\frac{1}{2}y+\sqrt{100-10y-\frac{3}{4}yy}$ for the firſt Number: Take away this and $y$ from 20, and there remains $\frac{yy}{x}=10-\frac{1}{2}y-\sqrt{100-10y-\frac{3}{4}yy}$ the third Number. And the Sum of the Squares ariſing from theſe 3 Numbers is $400-40y$, and ſo $400-40y=140$, or $y=6\frac{1}{2}$. And having found the mean Number $6\frac{1}{2}$, ſubſtitute it for $y$ in the firſt and third Number above found; and the firſt will become $6\frac{3}{4}+\sqrt{3\frac{5}{16}}$, and the third $6\frac{3}{4}-\sqrt{3\frac{5}{16}}$, as before.

PROBLEM XIV. To find 4 Numbers in continual Proportion, the 2 Means whereof together make 12, and the 2 Extremes 20.

Let $x$ be the ſecond Number, and $12-x$ will be the third, $\frac{xx}{12-x}$ the firſt; and $\frac{144-24x+xx}{x}$ the fourth; and conſequently $\frac{xx}{12-x}+\frac{144-24x+xx}{x}=20$. And by Reduction $xx=12x-30\frac{6}{7}$, or $x=6+\sqrt{5\frac{1}{7}}$. Which being found, the other Numbers are given from thoſe above.

PROBLEM XV. To find 4 Numbers continually proportional, whereof the Sum $a$ is given, and [alſo] the Sum of their Squares $b$.

Altho' we ought for the moſt Part to ſeek the Quantities requir'd immediately, yet if there are 2 that are ambiguous, that is, that involve both the ſame Conditions, as here the 2 Means and 2 Extremes of the 4 Proportionals) the beſt Way is to ſeek other Quantities that are not ambiguous, by which theſe may be determin'd, as ſuppoſe their Sum, or Difference, or Rectangle. Let us therefore make the Sum of the 2 mean Numbers to be $s$, and the Rectangle $r$, and the Sum of the Extremes will be $a-s$, and the Rectangle $r$, becauſe of the Proportionality. Now that from hence theſe 4 Numbers may be found, make $x$ the firſt, and $y$ the ſecond, and $s-y$ will be the third, and $a-s-x$ the fourth, and the Rectangle under the Means $sy-yy=r$, and thence one Mean $y=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-r}$, the other $s-y=\frac{1}{2}s-\sqrt{\frac{1}{4}ss-r}$. Alſo, the Rectangle under the Extremes $ax-sx-xx=r$, and thence one Extreme $x=\frac{a-s}{2}+\sqrt{\frac{ss-2as+aa}{4}-r}$, and the other $a-s-x=\frac{a-s}{2}-\sqrt{\frac{ss-2as+aa}{4}-r}$.

The Sum of the Squares of theſe 4 Numbers is $2ss-2as+aa-4r$, which is $=b$. Therefore $r=\frac{1}{2}ss-\frac{1}{2}as+\frac{1}{4}aa-\frac{1}{4}b$, which being ſubſtituted for $r$, there come out the 4 Numbers as follows:

$$\text{The 2 Means}\begin{cases}\frac{1}{2}s+\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}\\ \frac{1}{2}s-\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}.\end{cases}$$

$$\text{The 2 Extremes}\begin{cases}\frac{a-s}{2}+\sqrt{\frac{1}{4}b-\frac{1}{4}ss}\\ \frac{a-s}{2}-\sqrt{\frac{1}{4}b-\frac{1}{4}ss}\end{cases}$$

Yet there remains the Value of $s$ to be found. Wherefore, to abbreviate the Terms, for theſe Quantities ſubſtitute, and the 4 Proportionals will be

$$\begin{gathered}\frac{a-s}{2}+q\\ \frac{1}{2}s+p\\ \frac{1}{2}s-p\\ \frac{a-s}{2}-q\end{gathered}$$

and make the Rectangle under the ſecond and fourth equal to the Square of the third, ſince this Condition of the Queſtion is not yet ſatisfy'd, and you'll have $\frac{as-ss}{4}-\frac{1}{2}qs+\frac{pa-ps}{2}-pq=\frac{1}{4}ss-ps+pp$. Make alſo the Rectangle under the firſt and third equal to the Square of the ſecond, and you'll have $\frac{as-ss}{4}+\frac{1}{2}qs+\frac{-pa+ps}{2}-pq=\frac{1}{4}ss+ps+pp$. Take the firſt of theſe Æquations from the latter, and there will remain $qs-pa+ps=2ps$, or $qs=pa+ps$. Reſtore now $\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$ in the Place of $p$, and $\sqrt{\frac{1}{4}b-\frac{1}{4}ss}$ in the Place of $q$, and you'll have $s\sqrt{\frac{1}{4}b-\frac{1}{4}ss}=\overline{a+s}\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$, and by ſquaring $ss=-\frac{b}{a}s+\frac{1}{2}aa-\frac{1}{2}b$, or $s=-\frac{b}{2a}+\sqrt{\frac{bb}{4aa}+\frac{1}{2}aa-\frac{1}{2}b}$; which being found, the 4 Numbers ſought are given from what has been ſhewn above.

Problem XVI. If an annual Penſion of the [Number of] Pounds $a$, to be paid in the five next following Years, be bought for the ready Money $c$, to find what the Compound Intereſt of 100 *l. per Annum* will amount to?

Make $1-x$ the Compound Intereſt of the Money $x$ for a Year, that is, that the Money 1 to be paid after one Year is worth $x$ in ready Money; and, by Proportion, the Money $a$ to be paid after one Year will be worth $ax$ in ready Money, and after 2 Years [it will be worth] $axx$, and after 3 Years $ax^3$, and after 4 Years $ax^4$, and after 5 Years $ax^5$. Add theſe 5 Terms, and you'll have $ax^5+ax^4+ax^3+axx+ax=c$, or $x^5+x^4+x^3+x^2+x=\frac{c}{a}$ an Æquation of 5 Dimenſions, by Help of which when $x$ is found by the Rules to be taught hereafter, put $x:1::100:y$, and $y-100$ will be the Compound Intereſt of 100 *l. per Annum*.

It is [will be] ſufficient to have given theſe Inſtances in Queſtions where only the Proportions of Quantities are to be conſider'd, without the Poſitions of Lines: Let us now proceed to the Solutions of Geometrical Problems.

*How*

## *How Geometrical Queſtions may be reduc'd to Æquations.*

GEometrical Queſtions may be reduc'd ſometimes to Æquations with as much Eaſe, and by the ſame Laws, as thoſe we have propos'd concerning abſtracted Quantities. As if the right Line *AB* be to be cut [or divided] in mean and extreme Proportion [or Reaſon] in *C*, that is, ſo that *BE*, the Square of the greateſt Part, ſhall be equal to the Rectangle *BD* contain'd under the whole, and the leaſt Part; having put $AB = a$, and $BC = x$, then will $AC = a - x$, and $xx = a$ into $a - x$; an Æquation which by Reduction gives $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}aa}$. [*Vide Figure 6.*]

But in Geometrical [Caſes or] Affairs which more frequently occur, they ſo much depend on the various Poſitions and complex Relations of Lines, that they require ſome farther Invention and Artifice to bring them into Algebraick Terms. And tho' it is difficult to preſcribe any Thing in theſe Sorts of Caſes, and every Perſon's own Genius ought to be his Rule [or Guide] in theſe Operations; yet I'll endeavour to ſhew the Way to Learners. You are to know therefore, that Queſtions about the ſame Lines, related after any definite Manner to one another, may be variouſly propos'd, by making different Quantities the [*Quæſita*] or Things ſought, from different [*Data*] or Things given. But of what *Data* or *Quæſita* ſoever the Queſtion be propos'd, its Solution will follow the ſame Way by an Analytick Series, without any other Variation of Circumſtance beſides the feign'd Species of Lines, or the Names by which we are uſed to diſtinguiſh the given Quantities from thoſe ſought.

As if the Queſtion be of an *Iſoſceles CBD* inſcrib'd in a Circle, whoſe Sides *BC*, *BD*, and Baſe *CD*, are to be compar'd with the Diameter of the Circle *AB*. This may either be propos'd of the Inveſtigation of the Diameter from the given Sides and Baſe, or of the Inveſtigation of the Baſis from the given Sides and Diameter; or laſtly, of the Inveſtigation of the Sides from the given Baſe and Diameter; but however it be propos'd, it will be reduc'd to an Æquation by the ſame Series of an Analyſis. [*Vide Figure 7.*] *Viz.* If the Diameter be ſought, I put $AB = x$, $CD = a$, and *BC* or $BD = b$. Then (having drawn *AC*) by reaſon of the ſimilar Triangles $ABC$, and $CBE$, $AB$ will be to $BC :: BC : BE$, or $x : b :: b : BE$. Wherefore, $BE = \frac{bb}{x}$. Moreover, $CE$ is $= \frac{1}{2}CD$, or to $\frac{1}{2}a$; and by reaſon of the right Angle $CEB$, $CEq + BEq = BCq$, that is, $\frac{1}{4}aa + \frac{b^4}{xx} = bb$. Which Æquation, by Reduction, will give the Quantity $x$ ſought.

But if the Baſe be ſought, put $AB = c$, $CD = x$, and $BC$ or $BD = b$. Then ($AC$ being drawn) becauſe of the ſimilar Triangles $ABC$ and $CBE$, there is $AB : BC :: BC : BE$, or $c : b :: b : BE$. Wherefore $BE = \frac{bb}{c}$; and alſo $CE = \frac{1}{2}CD$, or $\frac{1}{2}x$. And becauſe the Angle $CEB$ is right, $CEq + BEq = BCq$, that is, $\frac{1}{4}xx + \frac{b^4}{cc} = bb$; an Æquation which will give by Reduction the ſought Quantity $x$.

But if the Side $BC$ or $BD$ be ſought, put $AB = c$, $CD = a$, and $BC$ or $BD = x$. And ($AC$ being drawn as before) by reaſon of the ſimilar Triangles $ABC$ and $CBE$, $AB$ is to $BC :: BC : BE$, or $c : x :: x : BE$. Wherefore $BE = \frac{xx}{c}$. Moreover, $CE$ is $= \frac{1}{2}CD$, or $\frac{1}{2}a$; and by reaſon of the right Angle $CEB$, $CEq + BEq = BCq$, that is, $\frac{1}{4}aa + \frac{x^4}{cc} = xx$; and the Æquation, by Reduction, will give the Quantity ſought, *viz.* $x$.

You ſee therefore that in every Caſe, the Calculus, by which you come to the Æquation, is the ſame every where, and brings out the ſame Æquation, excepting only that I have denoted the Lines by different Letters according as I made the *Data* and *Quæſita* [different]. And from different *Data* and *Quæſua* there ariſes a Diverſity in the Reduction of the Æquation found: For the Reduction of the Æquation $\frac{1}{4}aa + \frac{b^4}{xx} = bb$, in order to obtain $x = \frac{2bb}{\sqrt{4bb - aa}}$ the Value of $AB$, is different from the Reduction of the Æquation $\frac{1}{4}xx + \frac{b^4}{cc} = bb$, in order to obtain $x = \frac{2b}{c}\sqrt{cc - bb}$, the Value of $CD$; and the Reduction of the

Æquation $\frac{1}{4}aa + \frac{x^4}{cc} = xx$ very different to obtain $x = \sqrt{\frac{1}{4}cc \pm \frac{1}{2}c\sqrt{cc - aa}}$ the Value of *BC* or *BD*: (as well as this alſo, $\frac{1}{4}aa + \frac{b^4}{cc} = bb$, to bring out *c*, *a*, or *b*, ought to be reduc'd after different Methods) but there was no Difference in the Inveſtigation of theſe Æquations. And hence it is that [Analyſts] order us to make no Difference between the given and ſought Quantities. For ſince the ſame Computation agrees to any Caſe of the given and ſought Quantities, it is convenient that they ſhould be conceiv'd and compar'd without any Difference, that we may the more rightly judge of the Methods of computing them; or rather it is convenient that you ſhould imagine, that the Queſtion is propos'd of thoſe [*Data* and *Quæſita*] given and ſought Quantities, by which you think it is moſt eaſy for you to make out your Æquation.

Having therefore any Problem propos'd, compare the Quantities which it involves, and making no Difference between the given and ſought ones, conſider how they depend one upon another, that you may know what [Quantities] if they are aſſum'd, will, by proceeding Synthetically, give the reſt. To do which, there is no need that you ſhould at firſt of all conſider how they may be deduc'd from one another Algebraically; but this general Conſideration will ſuffice, that they may be ſome how or other deduc'd by a direct Connexion [with one another].

For Example, If the Queſtion be put of the Diameter of the Circle *AD*, and the three Lines *AB*, *BC*, and *CD* inſcrib'd in a Semi-circle, and from the reſt given you are to find *BC*; at firſt Sight it is manifeſt, that the Diameter *AD* determines the Semi-circle, and then, that the Lines *AB* and *CD* by Inſcription determine the Points *B* and *C*, and conſequently the Quantity ſought *BC*, and that by a direct Connexion; and yet after what Manner *BC* is to be had from theſe *Data* [or given Quantities] is not ſo evident to be found by an Analyſis. The ſame Thing is alſo to be underſtood of *AB* or *CD* if they were to be ſought from the other *Data*. [*Vide Figure* 8.] Now, if *AD* were to be found from the given Quantities *AB*, *BC*, and *CD*, it is equally evident it could not be done Synthetically; for the Diſtance of the Points *A* and *D* depends on the Angles

*B* and *C*, and thoſe Angles on the Circle in which the given Lines are to be inſcrib'd, and that Circle is not given without knowing the Diameter *AD*. The Nature of the Thing therefore requires, that *AD* be ſought, not Synthetically, but by aſſuming it [as given] to make thence a Regreſſion to the Quantities given.

When you ſhall have throughly perceiv'd the different Orderings [of the Proceſs] by which the Terms of the Queſtion may be explain'd, make Uſe of any of the Synthetical [Methods] by aſſuming Lines as given, from which you can form an eaſy Proceſs to others, tho' [the Regreſſion] to them may be very difficult. For the Computation, tho' it may proceed thro' various Mediums, yet will begin [originally] from thoſe [or ſuch] Lines; and will be ſooner perform'd by ſuppoſing the Queſtion to be ſuch, as if it was propos'd of thoſe *Data*, and ſome Quantity ſought that would eaſily come out from them, than by thinking of the Queſtion [in the Terms or Senſe] it is really propos'd in. Thus, in the propos'd Example, If from the reſt of the Quantities given you were to find *AD*: When I perceive that it cannot be done Synthetically, but yet that if it was done ſo, I could proceed in my Ratiocination on it in a direct Connexion [from one Thing] to others, I aſſume *AD* as given, and then I begin to compute as if it was given indeed, and ſome of the other Quantities, *viz.* ſome of the given ones, as *AB*, *BC*, or *CD*, were ſought. And by this Method, by carrying on the Computation from the Quantities aſſum'd after this Way to the others, as the Relations of the Lines [to one another] direct, there will always be obtain'd an Æquation between two Values of ſome one Quantity, whether one of thoſe Values be a Letter ſet down as a [Repreſentation or] Name at the Beginning of the Work for that Quantity, and the other a Value of it found out by Computation, or whether both be found by a Computation made after different Ways.

But when you have compar'd the Terms of the Queſtion thus generally, there is more Art and Invention requir'd to find out the particular Connexions or Relations of the Lines that ſhall accommodate them to [or render them fit for] Computation. For thoſe Things, which to a Perſon that does not ſo thoroughly conſider them, may ſeem to be immediately and by a very near Relation connected together, when we have a Mind to expreſs that Relation Algebraically, require a great deal more round-about Proceeding, and oblige you to begin your Schemes anew, and carry on your Computation Step by Step; as may appear by finding $BC$ from $AD$. $AB$, and $CD$. For you are only to proceed by ſuch Propoſitions or Enunciations that can fitly be repreſented in Algebraick Terms, whereof in particular you have ſome from [*Eucl.*] *Ax.* 19. *Prop.* 4. *Book* 6. and *Prop.* 47. of the firſt.

In the firſt Place therefore, the Calculus may be aſſiſted by the Addition and Subtraction of Lines, ſo that from the Values of the Parts you may find the Values of the whole, or from the Value of the whole and one of the Parts you may obtain the Value of the other Part.

In the ſecond Place, the Calculus is promoted by the Proportionality of Lines; for we ſuppoſe (as above) that the Rectangle of the mean Terms, divided by either of the Extremes, gives the Value of the other; or, which is the ſame Thing, if the Values of all four of the Proportionals are *firſt* had, we make an Equality [or Æquation] between the Rectangles of the Extremes and Means. But the Proportionality of Lines is beſt found out by the Similarity of Triangles, which, as it is known by the Equality of their Angles, the Analyſt ought in particular to be converſant in comparing them, and conſequently not to be ignorant of *Eucl. Prop.* 5, 13, 15, 29, and 32 of the firſt Book, and of *Prop.* 4, 5, 6, 7, and 8 of the ſixth Book, and of the 20, 21, 22, 27, and 31 of the third Book of his *Elem.* To which alſo may be added the 3d *Prop.* of the ſixth Book, wherein, from the Proportion of the Sides is inferr'd the Equality of the Angles, and *e contra.* Sometimes likewiſe the 36 and 37th *Prop.* of the third Book will do the ſame Thing.

In the third Place, [the Calculus] is promoted by the Addition or Subtraction of Squares, *viz.* In right angled Triangles we add the Squares of the leſſer Sides to obtain the Square of the greater, or from the Square of the greater Side we ſubtract the Square of one of the leſſer, to obtain the Square of the other

And on theſe few Foundations (if we add to them *Prop.* 1. of the 6th *Elem.* when the Buſineſs relates to Superficies, as alſo ſome Propoſitions taken out of the 11th and 12th of *Euclid*, when Solids come in Queſtion, the whole Analytick Art, as to right-lined Geometry, depends. Moreover, all the Difficulties of Problems may be reduc'd to the ſole Compoſition of Lines out of Parts, and the Similarity of Triangles; ſo that there is no Occaſion to make Uſe of other Theorems; becauſe they may all be reſolv'd into theſe two, and conſequently into the Solutions that may be drawn from them. And, for an Inſtance of this, I have ſubjoin'd a Problem about letting fall a Perpendicular upon the Baſe of an oblique-angled Triangle, [which is] ſolv'd without the Help of the 47th *Prop.* of the firſt Book of *Eucl.* But altho' it may be of [great] Uſe not to be ignorant of the moſt ſimple Principles on which the Solutions of Problems depend, and tho' by only their Help any [Problems] may be ſolv'd; yet, for Expedition ſake, it will be convenient not only that the 47th *Prop.* of the firſt Book of *Eucl.* whoſe Uſe is moſt frequent, but alſo that other Theorems ſhould ſometimes be made Uſe of.

As if [for Example] a Perpendicular being let fall upon the Baſe of an oblique angled Triangle, the Queſtion were (for the ſake of promoting Algebraick Calculus) to find the Segments of the Baſe; here it would be of Uſe to know, that the Difference of the Squares of the Sides is equal to the double Rectangle under the Baſe, and the Diſtance of the Perpendicular from the Middle of the Baſis.

If the Vertical Angle of any Triangle be biſected, it will not only be of Uſe to know, that the Baſe may be divided in Proportion to the Sides, but alſo, that the Difference of the Rectangles made by the Sides, and the Segments of the Baſe is equal to the Square of the Line that biſects the Angle.

If the Problem relate to Figures inſcrib'd in a Circle, this Theorem will frequently be of Uſe, *viz.* that in any quadrilateral Figure inſcrib'd in a Circle, the Rectangle of the Diagonals is equal to the Sum of the Rectangles of the oppoſite Sides.

The Analyſt may obſerve ſeveral Theorems of this Nature in his Practice, and reſerve them for his Uſe; but let him uſe them ſparingly, if he can, with equal Facility, or not much more Difficulty, hammer out the Solution from more ſimple Principles of Computation.

Wherefore let him take eſpecial Notice of the three Principles firſt propos'd, as being more known, more ſimple, more general, but a few, and yet ſufficient for all [Problems], and let him endeavour to reduce all Difficulties to them before others.

But that theſe Theorems may be accommodated to the Solution of Problems, the Schemes are oft times to be farther conſtructed, by producing out ſome of the Lines till they cut others, or become of an aſſign'd Length; or by drawing Lines parallel or perpendicular from ſome remarkable Point, or by conjoining ſome remarkable Points; as alſo ſometimes by conſtructing after other Methods, according as the State of the Problem, and the Theorems which are made Uſe of to ſolve it, ſhall require. As for Example, If two Lines that do not meet each other, make given Angles with a certain third Line, perhaps we produce them ſo, that when they concur, or meet, they ſhall form a Triangle, whoſe Angles, and conſequently the Reaſons of their Sides, ſhall be given; or, if any Angle is given, or be equal to any one, we often complete it into a Triangle given in Specie, or ſimilar to ſome other, and that by producing ſome of the Lines in the Scheme, or by drawing a Line ſubtending an Angle. If the Triangle be an oblique angled one, we often reſolve it into two right angled ones, by letting fall a Perpendicular. If the Buſineſs concerns multilateral [or many ſided] Figures, we reſolve them into Triangles, by drawing Diagonal Lines; and ſo in others; always aiming at this End, *viz.* that the Scheme may be reſolv'd either into given, or ſimilar, or right angled Triangles. Thus, in the Example propos'd, I draw the Diagonal *BD*, that the Trapezium *ABCD* may be reſolv'd into the two Triangles, *ABD* a right angled one, and *BDC* an oblique angled one. [*Vide Figure 9.*] Then I reſolve the oblique angled one into two right angled Triangles, by letting fall a Perpendicular from any of its Angles, *B C* or *D*, upon the oppoſite Side; as from *B* upon *CD* produc'd to *E*, that *BE* may meet it perpendicularly. But ſince the Angles *BAD* and *BCD* make in the mean while two right ones (by 22 *Prop.* 3 *Elem.*) as well as *BCE* and *BCD*, I perceive the Angles *BAD* and *BCE* to be equal; conſequently the Triangles *BCE* and *DAB* to be ſimilar. And ſo I ſee that the Computation (by aſſuming *AD*, *AB*, and *BC* as if *CD* were ſought) may be thus carry'd on, *viz.* *AD* and *AB* (by reaſon of the right angled Triangle *ABD*) give you *BD*. *AD*, *AB*, *BD*, and *BC* (by reaſon of the ſimilar Triangles *ABD* and *CEB*) give *BE* and *CE*. *BD* and *BE* (by reaſon of the right angled Triangle *BED*) give *ED*; and $ED - EC$ gives *CD*. Whence there will be obtain'd an Æquation between the Value of *CD* ſo found out, and the [ſmall Algebraick] Letter that denotes it. We may alſo (and for the greateſt Part it is better ſo to do, than to follow the Work too far in one continued Series) begin the

Computation from different Principles, or at leaſt promote it by divers Methods to any one and the ſame Concluſion, that at length there may be obtain'd two Values of any the ſame Quantity, which may be made equal to one another. Thus, $AD$, $AB$, and $BC$, give $BD$, $BE$, and $CE$ as before; then $CD+CE$ gives $ED$; and, laſtly, $BD$, and $ED$ (by reaſon of the right angled Triangle $BED$) give $BE$. You might alſo very well form the Computation thus, that the Values of thoſe Quantities ſhould be ſought between which any other known Relation interceeds, and then that Relation will bring it to an Æquation. Thus, ſince the Relation between the Lines $BD$, $DC$, $BC$, and $CE$, is manifeſt from the 12th *Prop.* of the ſecond Book of the *Elem. viz.* that $BDq - BCq - CDq = 2CD \times CE$: I ſeek $BDq$ from the aſſum'd $AD$ and $AB$; and $CE$ from the aſſum'd $AD$, $AB$, and $BC$. And, laſtly, aſſuming $CD$ I make $BDq - BCq - CDq = 2CD \times CE$. After ſuch Ways, and led by theſe Sorts of Conſultations, you ought always to take care of the Series of the Analyſis, and of the Scheme to be conſtructed in order to it, at once.

Hence, I believe, it will be manifeſt what Geometricians mean, when they bid you imagine that to be already done which is ſought. For making no Difference between the known and unknown Quantities, you may aſſume any of them to begin your Computation from, as much as if all had [indeed] been known by a previous Solution, and you were no longer to conſult the Solution of the Problem, but only the Proof of that Solution. Thus, in the firſt of the three Ways of computing already deſcribed, altho' perhaps $AD$ be really ſought, yet I imagine $CD$ to be the Quantity ſought, as if I had a mind to try whether its Value deriv'd from $AD$ will coincide with [or be equal to] its Quantity before known. So alſo in the two laſt Methods, I don't propoſe, as my Aim, any Quantity to be ſought, but only ſome how or other to bring out an Æquation from the Relations of the Lines: And, for ſake of that Buſineſs I aſſume all [the Lines] $AD$, $AB$, $BC$, and $CD$ as known, as much as if (the Queſtion being before ſolv'd) the Buſineſs was to enquire whether ſuch and ſuch Lines would ſatisfy the Conditions of it, by [falling in or] agreeing with any Æquations which the Relations of the Lines can exhibit. I enter'd upon the Buſineſs at firſt Sight after this Way, and with ſuch [Sort of] Conſultations; but when I arrive at an Æquation, I change my Method, and endeavour to find the Quantity ſought by the Reduction and Solution of that Æquation. Thus, laſtly, we aſſume often more Quantities as known, than what are expreſs'd in the State of the Queſtion. Of this you may ſee an eminent Example in the 42d of the following Problems, where I have aſſum'd $a$, $b$, and $c$, in the Æquation $a+bx+cx^2=yy$ for determining the Conick Section; as alſo the other Lines $r$, $s$, $t$, $v$, of which the Problem, as it is propos'd, hints nothing. For you may aſſume any Quantities by the Help whereof it is poſſible to come to Æquations; only taking this Care, that you obtain as many Æquations from them as you aſſume Quantities really unknown.

After you have conſulted your Method of Computation, and drawn up your Scheme, give Names to the Quantities that enter into the Computation, (that is, from which being aſſum'd the Values of others are to be deriv'd, till at laſt you come to an Æquation) chuſing ſuch as involve all the Conditions of the Problem, and ſeem accommodated before others to the Buſineſs, and that ſhall render the Concluſion (as far as you can gueſs) more ſimple, but yet not more than what ſhall be ſufficient for your Purpoſe. Wherefore, don't give proper Names to Quantities which may be denominated from Names already given. Thus, from a whole Line given and its Parts, from the three Sides of a right angled Triangle, and from three or four Proportionals, ſome one of the leaſt conſiderable we leave without a Name, becauſe its Value may be deriv'd from the Names of the reſt. As in the Example already brought, if I make $AD=x$; and $AB=a$, I denote $BD$ by no Letter, becauſe it is the third Side of a right angled Triangle $ABD$, and conſequently its Value is $\sqrt{xx-aa}$. Then if I ſay $BC=b$, ſince the Triangles $DAB$ and $BCE$ are ſimilar, and thence the Lines $AD:AB::BC:CE$ proportional, to three whereof, *viz.* to $AD$, $AB$, and $BC$ there are already Names given; for that reaſon I leave the fourth $CE$ without a Name, and in its room I make Uſe of $\frac{ab}{x}$ diſcover'd from the foregoing Proportionality. And ſo if $DC$ be called $c$, I give no Name to $DE$, becauſe from its Parts, $DC$ and $CE$, or $c$ and $\frac{ab}{x}$, its Value $c+\frac{ab}{x}$ comes out. [*Vide Figure* 10.]

But while I am talking of theſe Things, the Problem is almoſt reduc'd to an Æquation. For, after the aforeſaid Letters are ſet down for the Species of the principal Lines, there remains nothing elſe to be done, but that out of thoſe Species the Values of other Lines be made out according to a preconceiv'd Method, till after ſome foreſeen Way they come to an Æquation. And I can ſee nothing wanting in this Caſe, except that by [means of] the right angled Triangles $BCE$ and $BDE$ I can bring out a double Value of $BE$, viz. $BCq - CEq$ $\left(\text{or } bb - \frac{aabb}{xx}\right) = BEq$; as alſo $BDq - DEq$ $\left(\text{or } xx - aa - cc - \frac{2abc}{x} - \frac{aabb}{xx}\right) = BEq$. And hence (blotting out on both Sides $\left.\frac{aabb}{xx}\right)$ I ſhall have the Æquation $bb = xx - aa - cc - \frac{2abc}{x}$; which being reduc'd, becomes $x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc$.

But ſince I have reckon'd up ſeveral Methods for the Solution of this Problem, and thoſe not much unlike [one another] in the precedent [Paragraphs], of which that taken from *Prop.* 12. of the ſecond Book of the *Elem.* being ſomething cleverer than the reſt, we will here ſubjoin it. Make therefore $AD = x$, $AB = a$, $BC = b$, and $CD = c$, and you'll have $BDq = xx - aa$, and $CE = \frac{ab}{x}$ as before. Theſe Species therefore being ſubſtituted in the Theorem for $BDq - BCq - CDq = 2CD \times CE$, there will ariſe $xx - aa - bb - cc = \frac{2abc}{x}$, and after Reduction, $x^3 = \begin{matrix} aa \\ +bb \\ +cc \end{matrix} x + 2abc$, as before.

But that it may appear how great a Variety there is in the Invention of Solutions, and that it is not very difficult for a prudent Geometrician to light upon them, I have thought fit to teach [or ſhew] other Ways of doing the ſame Thing. And having drawn the Diagonal $BD$, if in room of the Perpendicular $BE$, which before was let fall from the Point $B$ upon the Side $DC$, you now let fall a Perpendicular from the Point $D$ upon the Side $BC$, or from the

the Point *C* upon the Side *BD*, by which the oblique angled Triangle *BCD* may any how be reſolv'd into two right angled Triangles, you may come almoſt by the ſame Methods I have already deſcrib'd to an Æquation. And there are other Methods very different from theſe; as if there are drawn two Diagonals, *AC* and *BD*, *BD* will be given by aſſuming *AD* and *AB*; as alſo *AC* by aſſuming *AD* and *CD*; then, by the known Theorem of Quadrilateral Figures inſcrib'd in a Circle, *viz.* That $AD \times BC + AB \times CD$ is $= AC \times BD$, you'll obtain an Æquation. [*Vide Figure* 11.] Suppoſe therefore remaining the Names of the Lines *AD*, *AB*, *BC*, *CD*, *viz.* $x, a, b, c$, *BD* will be $= \sqrt{xx - aa}$, and $AC = \sqrt{xx - cc}$, by the 47th *Prop.* of the firſt *Elem.* and theſe Species of the Lines being ſubſtituted in the Theorem we juſt now mention'd, there will come out $xb + ac = \sqrt{xx - cc} \times \sqrt{xx - aa}$. The Parts of which Æquation being ſquar'd and reduc'd, you'll again have $x^3 = \begin{matrix} aa \\ +bb \\ +cc \end{matrix}\, x + 2abc$.

But, moreover, that it may be manifeſt after what Manner the Solutions drawn from that Theorem may be thence reduc'd to only the Similarity of Triangles, erect *BH* perpendicular to *BC*, and meeting *AC* in *H*, and there will be form'd the Triangles *BCH*, *BDA* ſimilar, by reaſon of the right Angles at *B*, and equal Angles at *C* and *D*, (by the 21. 3. *Elem.*) as alſo the Triangles *BCD*, *BHA* [which are alſo] ſimilar, by reaſon of the equal Angles both at *B*, (as may appear by taking away the common Angle *DBH* from the two right ones) as alſo at *D* and *A* (by 21. 3 *Elem.*) You may ſee therefore, that from the Proportionality $BD : AD :: BC : HC$, there is given the Line *HC*; as alſo *AH* from the Proportionality $BD : CB :: AB : AH$. Whence ſince $AH + HC = AC$, you have an Æquation. The Names therefore aforeſaid of the Lines remaining, *viz.* $x, a, b, c$, as alſo the Values of the Lines *AC* and *BD*, *viz.* $\sqrt{xx - cc}$ and $\sqrt{xx - aa}$, the firſt Proportionality will give $HC = \frac{xb}{\sqrt{xx - aa}}$, and the ſecond will give $AH = \frac{ca}{\sqrt{xx - aa}}$. Whence, by reaſon of $AH$

$+HC=AC$, you'll have $\frac{bx+ac}{\sqrt{xx-aa}}=\sqrt{xx-cc}$; an Æquation which (by multiplying by $\sqrt{xx-aa}$, and by ſquaring) will be reduc'd to a Form often deſcrib'd in the preceding Pages.

But that it may yet farther appear what a Plenty of Solutions may be found, produce $BC$ and $AD$ till they meet in $F$, and the Triangles $ABF$ and $CDF$ will be ſimilar, becauſe the Angle at $F$ is common, and the Angles $ABF$ and $CDF$ (while they compleat the Angle $CDA$ to two right ones, by 13, 1. and 22, 3 *Elem.*) are equal. [*Vide Figure* 12.] Wherefore, if beſides the four Terms which compoſe the Queſtion there was given $AF$, the Proportion $AB : AF :: CD : CF$ would give $CF$. Alſo $AF-AD$ would give $DF$, and the Proportion $CD : DF :: AB : BF$ would give $BF$; whence (ſince $BF-CF=BC$) there would ariſe an Æquation. But ſince there are aſſum'd two unknown Quantities as if they were given, there remains another Æquation to be found. I let fall therefore $BG$ at right Angles upon $AF$, and the Proportion $AD : AB :: AB : AG$ will give $AG$; which being had, the Theorem in the 13, 2 *Eucl.* *viz.* that $BFq+2FAG=ABq+AFq$ will give another Æquation. $a$, $b$, $c$, $x$ remaining therefore as before, and making $AF=y$, you'll have (by inſiſting on the Steps already laid down) $\frac{cy}{a}=CF$. $\overline{y-x}=DF$. $\frac{\overline{y-x}\times a}{c}=BF$. And thence $\frac{\overline{y-x}\times a}{c}-\frac{cy}{a}=b$, the firſt Æquation. Alſo $\frac{aa}{x}$ will be $=AG$, and conſequently $\frac{aayy-2a^2xy+a^2x^2}{cc}+\frac{2aay}{x}=aa+yy$ for the ſecond Æquation. Which two, by Reduction, will give the Æquation ſought, *viz.* The Value of $y$ found by the firſt Æquation is $\frac{abc+aax}{aa-cc}$, which being ſubſtituted in the ſecond, will give an Æquation, from which rightly order'd will come out $x^3=\begin{matrix}aa\\+bb\\+cc\end{matrix}x+2abc$, as before.

And ſo, if *AB* and *DC* are produc'd till they meet one another, the Solution will be much the ſame, unleſs perhaps it be ſomething eaſier. Wherefore I will ſubjoyn another Specimen of this [Problem] from a Fountain very unlike the former, *viz.* by ſeeking the Area of the Quadrilateral Figure propos'd, and that doubly. I draw therefore the Diagonal *BD*, that the Quadrilateral Figure may be reſolv'd into two Triangles. Then uſing the Names of the Lines $x$, $a$, $b$, $c$, as before, I find $BD = \sqrt{xx-aa}$, and $\frac{1}{2}a\sqrt{xx-aa}$ $(=\frac{1}{2}AB \times BD)$ the Area of the Triangle *ABD*. Moreover, having let fall *BE* perpendicularly upon *CD* you'll have (by reaſon of the ſimilar Triangles *ABD*, *BCE*) $AD : BD :: BC : BE$, and conſequently $BE = \frac{b}{x}\sqrt{xx-aa}$. Wherefore alſo $\frac{bc}{2x}\sqrt{xx-aa}$ $(=\frac{1}{2}CD \times BE)$ will be the Area of the Triangle *BCD*. Now, by adding theſe Area's, there will ariſe $\frac{ax+bc}{2x}\sqrt{xx-aa}$, the Area of the whole Quadrilateral. After the ſame Way, by drawing the Diagonal *AC*, and ſeeking the Area's of the Triangles *ACD* and *ACB*, and adding them, there will again be obtain'd the Area of the Quadrilateral Figure $\frac{cx+ba}{2x}\sqrt{xx-cc}$. Wherefore, by making theſe Area's equal, and multiplying both by $2x$, you'll have $\overline{ax+bc}\sqrt{xx-aa} = \overline{cx+ba}\sqrt{xx-cc}$, an Æquation which, by ſquaring and dividing by $aax-ccx$, will be reduc'd to the Form already often found out, $x^3 \begin{array}{l} aa \\ +bb \\ +cc \end{array} x + 2abc$.

Hence it may appear how great a Plenty of Solutions may be had, and that ſome Ways are much more neat than others. Wherefore, if the Method you take from your firſt Thoughts, for ſolving a Problem, be but ill accommodated to Computation, you muſt again conſider the Relations of the Lines, till you ſhall have hit on a Way as fit and elegant as poſſible. For thoſe Ways that offer themſelves at firſt Sight, may create ſufficient Trouble, perhaps, if they are made Uſe of. Thus, in the Problem we have been upon, nothing would have been more difficult than to have fallen upon the following Method inſtead of one of the precedent ones. [*Vide Figure* 13.] Having let fall $BR$ and $CS$ perpendicular to $AD$, as alſo $CT$ to $BR$, the Figure will be reſolv'd into right angled Triangles. And it may be ſeen, that $AD$ and $AB$ give $AR$, $AD$ and $CD$ give $SD$, $AD - AR - SD$ gives $RS$ or $TC$. Alſo $AB$ and $AR$ give $BR$, $CD$ and $SD$ give $CS$ or $TR$, and $BR - TR$ gives $BT$. Laſtly, $BT$ and $TC$ give $BC$, whence an Æquation will be obtain'd. But if any one ſhould go to compute after this Rate, he would fall into larger [and more perplex'd] Algebraick Terms than are any of the former, and more difficult to be brought to a final Æquation.

So much for the Solution of Problems in right lined Geometry; unleſs it may perhaps be worth while to note moreover, that when Angles, or Poſitions of Lines expreſs'd by Angles, enter the State of the Queſtion, Lines, or the Proportions of Lines ought to be uſed inſtead of Angles, *viz.* ſuch as may be deriv'd from given Angles by a Trigonometrical Calculation; or from which being found, the Angles ſought [will] come out by the ſame Calculus. Several Inſtances of which may be ſeen in the following Pages.

As for what belongs to the Geometry of Curve Lines, we uſe to denote them, either by deſcribing them by the local Motion of right Lines, or by uſing Æquations indefinitely expreſſing the Relation of right Lines diſpos'd [in order] according to ſome certain Law, and ending at the Curve Lines. The Antients did the ſame by the Sections of Solids, but leſs commodiouſly. But the Computations that regard Curves deſcrib'd after the firſt Way, are no otherwiſe perform'd than in the precedent [Pages.] As if $AKC$ be a Curve Line deſcrib'd by $K$ the Vertical Point of the Square $AK\varphi$, whereof one Leg $AK$ freely ſlides through the Point $A$ given by Poſition, while the other $K\varphi$ of a determinate Length is carry'd along the right Line $AD$ alſo given by Poſition, and you are to find the Point $C$ in which any right Line $CD$ given [alſo] by Poſition ſhall cut this Curve: I draw the right Lines $AC$, $CF$, which may repreſent the Square in the Poſition ſought, and the Relation of the Lines (without any Difference [or Regard] of what is given or ſought, or any Reſpect had to the Curve) being conſider'd, I perceive the Dependency of the others upon $CF$ and any of theſe four, *viz.* $BC$, $BF$, $AF$, and $AC$ to be Synthetical, two whereof I therefore aſſume, as $CF = a$, and

$CB = x$, and beginning the Computation from thence, I preſently obtain $BF = \sqrt{aa - xx}$, and $AB = \frac{xx}{\sqrt{aa - xx}}$, by reaſon of the right Angle $CBF$, and that the Lines $BF : BC :: BC : AB$ are continual Proportionals. Moreover, from the given Poſition of $CD$, $AD$ is given, which I therefore call $b$, there is alſo given the reaſon of $BC$ to $BD$, which I make as $d$ to $e$, and you have $BD = \frac{ex}{d}$, and $AB = b - \frac{ex}{d}$. [*Vide Figure* 14.] Therefore $b - \frac{ex}{d} = \frac{xx}{\sqrt{aa - xx}}$, an Æquation which (by ſquaring its Parts and multiplying by $aa - xx$) will be reduc'd to this Form,

$$x^4 = \frac{2bdex^3 \begin{matrix} -bbdd \\ +aaee \end{matrix} xx - 2aabdex + aabbdd.}{dd + ee}$$

Whence, laſtly, from the given Quantities $a$, $b$, $d$, and $e$, $x$ may to be found by Rules hereafter to be given, and at that Interval [or Diſtance] $x$ or $BC$, a right Line drawn parallel to $AD$ will cut $CD$ in the Point ſought $C$.

Now, if we don't uſe Geometrical Deſcriptions but Æquations to denote the Curve Lines by, the Computations will thereby become as much ſhorter and eaſier, as the gaining of thoſe Æquations can make them. As if the Interſection $C$ of the given Ellipſis $ACE$ with the right Line $CD$ given by Poſition, be ſought. To denote the Ellipſis, I take ſome known Æquation proper to it, as $rx - \frac{r}{q}xx = yy$, where $x$ is indefinitely put for any Part of the Axis $Ab$ or $AB$, and $y$ for the Perpendicular $bc$ or $BC$ terminated at the Curve; and $r$ and $q$ are given from the given Species of the Ellipſis. [*Vide Figure* 15.] Since therefore $CD$ is given by Poſition, $AD$ will be alſo given, which call $a$; and $BD$ will be $a - x$; alſo the Angle $ADC$ will be given, and thence the Reaſon of $BD$ to $BC$, which call 1 to $e$, and $BC$ $(y)$ will be $= ea - ex$, whoſe Square $eeaa - 2eeax + eexx$ will be equal to $rx - \frac{r}{q}xx$. And thence by Reduction there will ariſe $xx = \frac{2aeex + rx - aaee}{ee + \frac{r}{q}}$,

or $x = \frac{aee + \frac{1}{2}r \pm e\sqrt{ar + \frac{rr}{4ee} - \frac{aar}{q}}}{ee + \frac{r}{q}}$.

Moreover, altho' a Curve be denoted by a Geometrical Deſcription, or by a Section of a Solid, yet thence an Æquation may be obtain'd, which ſhall define the Nature of the Curve, and conſequently all the Difficulties of Problems propos'd about it may be reduc'd hither.

Thus, in the former Example, if *AB* be called $x$, and *BC* $y$, the third Proportional *BF* will be $\frac{yy}{x}$, whoſe Square, together with the Square of *BC*, is equal to *CFq*, that is, $\frac{y^4}{xx} + yy = aa$; or $y^4 + xxyy = aaxx$. And this is an Æquation by which every Point *C* of the Curve *AKC*, agreeing or correſponding to any Length of the Baſe (and conſequently the Curve it ſelf) is defin'd, and from whence conſequently you may obtain the Solutions of Problems propos'd concerning this Curve.

After the ſame Manner almoſt, when a Curve is not given in Specie, but propos'd to be determin'd, you may feign an Æquation at Pleaſure, that may generally contain its Nature, and aſſume this to denote it as if it was given, that from its Aſſumption you can any Way come to Æquations by which the Aſſumptions may be determin'd; Examples whereof you have in ſome of the following Problems, which I have collected for a more full Illuſtration of this Doctrine, and which I now proceed to deliver.

## PROBLEM I.

*Having a finite right Line BC given, from whoſe Ends the two right Lines BA, CA are drawn with the given Angles ABC, ACB; to find AD the Height of the Concourſe A,* [*or the Point of their Meeting*] *above the given Line BC.* [Vide Figure 16.]

MAke $Bc = a$, and $AD = y$; and ſince the Angle $ABD$ is given, there will be given (from the Table of Sines or Tangents) the Ratio between the Lines $AD$ and $BD$ which make as $d$ to $e$. Therefore $d : e :: AD\ (y) : BD$. Wherefore $BD = \frac{ey}{d}$. In like Manner, by reaſon of the given Angle $ACD$ there will be given the Ratio between $AD$ and $DC$, which make as $d$ to $f$, and you'll have $DC = \frac{fy}{d}$. But $BD + DC = BC$, that is, $\frac{ey}{d} + \frac{fy}{d} = a$. Which reduc'd by multiplying both Parts of the Æquation by $d$, and dividing by $e + f$, becomes $y = \frac{ad}{e + f}$.

## PROBLEM II.

*The Sides AB, AC of the Triangle ABC being given, and alſo the Baſe BC, which the Perpendicular AD* [*let fall*] *from the Vertical Angle cuts in D, to find the Segments BD and DC.* [Vide Figure 17.]

LET $AB = a$, $AC = b$, $BC = c$, and $BD = x$, and $DC$ will $= c - x$. Now ſince $ABq - BDq\ (aa - xx) = ADq$; and $ACq - DCq\ (bb - cc + 2cx - xx) = ADq$; you'll have $aa - xx = bb - cc + 2cx - xx$; which by Reduction becomes $\frac{aa - bb + cc}{2c} = x$.

But that it may appear that all the Difficulties of all Problems may be reſolv'd by only the Proportionality of Lines, without the Help of the 47 of 1 *Eucl.* altho' not without round-about Methods, I thought fit to ſubjoyn the following Solution of this Problem over and above. From the Point $D$ let fall the Perpendicular $DE$ upon the Side $AB$, and the Names of the Lines, already given, remaining, you'll have $AB : BD :: BD : BE$.

$a : x :: x . \frac{xx}{a}$. And $BA - BE$ $\left(a - \frac{xx}{a}\right) = EA$, alſo $EA : AD :: AD : AB$, and conſequently $EA \times AB$ $(aa - xx)$ $= ADq$. And ſo, by reaſoning about the Triangle $ACD$, there will be found again $ADq = bb - cc + 2cx - xx$. Whence you will obtain as before $x = \frac{aa - bb + cc}{2c}$.

## PROBLEM III.

### *The Area and Perimeter of the right angled Triangle ABC being given, to find the Hypothenuſe BC.* [Vide Figure 18.]

LET the Perimeter be [called] $a$, the Area $bb$, make $BC = x$, and $AC = y$; then will $AB = \sqrt{xx - yy}$; whence again the Perimeter $(BC + AC + AB)$ is $x + y + \sqrt{xx - yy}$, and the Area $(\frac{1}{2}AC \times AB)$ is $\frac{1}{2}y\sqrt{xx - yy} = bb$. Therefore $x + y + \sqrt{xx - yy} = a$, and $\frac{1}{2}y\sqrt{xx - yy} = bb$.

The latter of theſe Æquations gives $\sqrt{xx - yy} = \frac{2bb}{y}$; wherefore I write $\frac{2bb}{y}$ for $\sqrt{xx - yy}$ in the former Æquation, that the Aſymmetry may be taken away; and there comes out $x + y + \frac{2bb}{y} = a$, or multiplying by $y$, and ordering [the Æquation] $yy = ay - xy - 2bb$. Moreover, from the Parts of the former Æquation I take away $x + y$, and there remains $\sqrt{xx - yy} = a - x - y$, and ſquaring the Parts to take away again the Aſymmetry, there comes out $xx - yy = aa - 2ax - 2ay + xx + 2xy + yy$, which order'd and divided by 2, becomes $yy = ay - xy$

$+ aa$

$+ax-\frac{1}{2}aa$. Laſtly, making an Equality between the 2 Values of $yy$, I have $ay-xy-2bb=ay-xy+ax-\frac{1}{2}aa$, which reduc'd becomes $\frac{1}{2}a-\frac{2bb}{a}=x$.

### *The ſame otherwiſe.*

Let ½ the Perimeter $=a$, the Area $=bb$, and $BC=x$, and $AC+AB=2a-x$. Now ſince $xx$ $(BCq)$ is $=ACq+ABq$, and $4bb=2AC\times AB$, $xx+4bb$ will $=ACq+ABq+2AC\times AB=$ to the Square of $\overline{AC+AB}$ $=$ to the Square of $\overline{2a-x}=4aa-4ax+xx$. That is, $xx+4bb=4aa-4ax+xx$, which reduc'd becomes $a-\frac{bb}{a}=x$.

## PROBLEM IV.

### *Having given the Area, Perimeter, and one of the Angles A of any Triangle ABC, to determine the reſt.* [Vide Figure 19.]

LET the Perimeter be $=a$, and the Area $=bb$, and from either of the unknown Angles, as $C$, let fall the Perpendicular $CD$ to the oppoſite Side $AB$; and by reaſon of the given Angle $A$, $AC$ will be to $CD$ in a given Ratio, ſuppoſe as $d$ to $e$. Call therefore $AC=x$, and $CD$ will $=\frac{ex}{d}$, by which divide the double Area, and there will come out $\frac{2bbd}{ex}=AB$. Add $AD$ (viz. $\sqrt{ACq-CDq}$, or $\frac{x}{d}\times\sqrt{dd-ee}$) and there will come out $BD=\frac{2bbd}{ex}+\frac{x}{d}\times\sqrt{dd-ee}$, to the Square whereof add $CDq$, and there will ariſe $\frac{4b^4dd}{eexx}+xx+\frac{4bb}{e}\sqrt{dd-ee}=BCq$. Moreover, from the Perimeter take away $AC$ and $AB$, and there will remain $a-x-\frac{2bbd}{ex}=BC$, the Square whereof $aa-2ax+xx-\frac{4abbd}{ex}+\frac{4bbd}{e}+\frac{4b^4dd}{eexx}$ make equal to the Square before found; and neglecting the Equivalents, you'll have $\frac{4bb}{e}\sqrt{dd-ee}=aa-2ax-\frac{4abbd}{ex}+\frac{4bbd}{e}$. And this, by aſſuming $4af$ for the given Terms $aa+\frac{4bbd}{e}-\frac{4bb}{e}\sqrt{dd-ee}$; and by reducing becomes $xx=2fx-\frac{2bbd}{e}$, or $x=f\pm\sqrt{ff-\frac{2bbd}{e}}$.

The ſame Æquation would have come out alſo by ſeeking the Leg $AB$; for the Sides $AB$ and $AC$ are indifferently alike to all the Conditions of the Problem. Wherefore if $AC$ be made $=f-\sqrt{ff-\frac{2bbd}{e}}$, $AB$ will $=f+\sqrt{ff-\frac{2bbd}{e}}$, and reciprocally; and the Sum of theſe $2f$ ſubtracted from the Perimeter, leaves the third Side $BC=a-2f$.

## PROBLEM V.

### *Having given the Altitude, Baſe, and Sum of the Sides, to find the Triangle.*

LET the Altitude $CD=a$, half the Baſis $AB=b$, half the Sum of the Sides $=c$, and their Semi-difference $=z$; and the greater Side as $BC=c+z$, and the leſſer $AC=c-z$. Subtract $CDq$ from $CBq$, and alſo from $ACq$, and hence will $BD=\sqrt{cc+2cz+zz-aa}$, and thence $AD=\sqrt{cc-2cz+zz-aa}$. Subtract alſo $AB$ from $BD$, and $AD$ will again $=\sqrt{cc+2cz+zz-aa}-2b$. Having now ſquared the Values of $AD$, and order'd the Terms, there will ariſe $bb+cz=b\sqrt{cc+2cz+zz-aa}$. Again, by ſquaring and reducing into Order, you'll obtain $cczz-bbzz=bbcc-bbaa-b^4$. And $z=b\sqrt{1-\frac{aa}{cc-bb}}$. Whence the Sides are given.

## PROBLEM VI.

*Having given the Baſe AB, and the Sum of the Sides AC + BC, and alſo the Vertical Angle C, to determine the Sides.* [Vide Figure 20.]

MAke the Baſe $= a$, half the Sum of the Sides $= b$, and half the Difference $= x$, and the greater Side $BC$ will be $= b + x$, and the leſſer $AC = b - x$. From either of the unknown Angles $A$ let fall the Perpendicular $AD$ to the oppoſite Side $BC$, and by reaſon of the given Angle $C$ there will be given the Ratio of $AC$ to $CD$, ſuppoſe as $d$ to $e$; and then $CD$ will $= \frac{eb - ex}{d}$. Alſo, by 11. 2 *Elem.* $\frac{ACq - ABq + BCq}{2BC}$, that is, $\frac{2bb + 2xx - aa}{2b + 2x} = CD$; and ſo you have an Æquation between the Values of $CD$. And this reduc'd, $x$ becomes $= \sqrt{\frac{daa + 2ebb - 2dbb}{2d + 2e}}$. whence the Sides are given.

If the Angles at the Baſe were ſought, the Concluſion would be more neat, as draw $EC$ biſecting the given Angle and meeting the Baſe in $E$; and $AB : AC + BC$ (:: $AE$ : $AC$) :: Sine Angle $ACE$ : Sine Angle $AEC$; and if from the Angle $AEC$, and alſo from its Complement $BEC$ you ſubtract $\frac{1}{2}$ the Angle $C$, there will be left the Angles $ABC$ and $BAC$.

## PROBLEM VII.

*Having given the Sides of any Parallelogram AB, BD, DC, and AC, and one of the Diagonals BC, to find the other Diagonal AD.* [Vide Figure 21.]

LET $E$ be the Concourſe of the Diagonals, and to the Diagonal $BC$ let fall the Perpendicular $AF$, and by the 13. 2 *Elem.* $\frac{ACq - ABq + BCq}{2BC} = CF$. And alſo

$\frac{ACq - AEq + ECq}{2EC} = CF$. Wherefore ſince $EC = \frac{1}{2}BC$, and $AE = \frac{1}{2}AD$, $\frac{ACq - ABq + BCq}{2BC} = \frac{ACq - \frac{1}{4}ADq + \frac{1}{4}BCq}{BC}$, and having reduc'd, $AD = \sqrt{2ACq + 2ABq - BCq}$.

Whence, by the by, in any Parallelogram, the Sum of the Squares of the Sides is equal to the Sum of the Squares of the Diagonals.

## PROBLEM VIII.

*Having given the Angles of the Trapezium ABCD, alſo its Perimeter and Area, to determine the Sides.* [Vide Figure 22.]

PRoduce any two of the Sides $AB$ and $DC$ till they meet in $E$, and let $AB = x$, and $BC = y$, and becauſe all the Angles are given, there are given the Ratio's of $BC$ to $CE$ and $BE$, which make $d$ to $e$ and $f$; and $CE$ will be $= \frac{ey}{d}$, and $BE = \frac{fy}{d}$, and conſequently $AE = x + \frac{fy}{d}$. There are alſo given the Ratio's of $AE$ to $AD$, and of $AE$ to $DE$; which make as $g$ to $d$, and as $h$ to $d$; and $AD$ will $= \frac{dx + fy}{g}$, and $ED = \frac{dx + fy}{h}$, and conſequently $CD = \frac{dx + fy}{h} - \frac{ey}{d}$, and the Sum of all the Sides $x + y + \frac{dx + fy}{g} + \frac{dx + fy}{h} - \frac{ey}{d}$; which, ſince it is given, call it $a$, and the Terms will be abbreviated by writing $\frac{p}{r}$ for the given $1 + \frac{d}{g} + \frac{d}{h}$, and $\frac{q}{r}$ for the given $1 + \frac{f}{g} + \frac{f}{h} - \frac{e}{d}$, and you'll have the Æquation $\frac{px + qy}{r} = a$.

Moreover, by Reaſon of all the Angles given, there is given the [Ratio or] Reaſon of $BCq$ to the Triangle $BCE$, which make as $m$ to $n$, and the Triangle $BCE = \frac{n}{m}yy$. There is alſo given the Ratio of $AEq$ to the Triangle $ADE$;

Tab. I. pag. 106

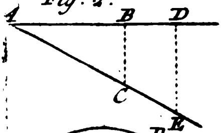


Fig. 2.

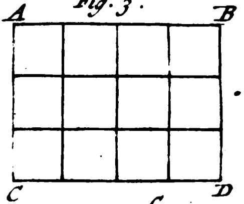


Fig. 3.

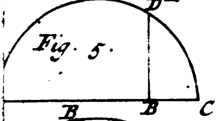
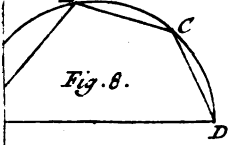


Fig. 5.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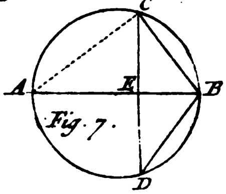
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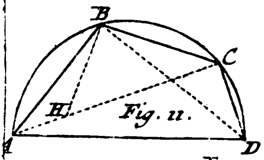
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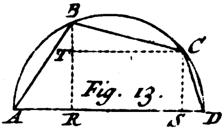


Fig. 13.

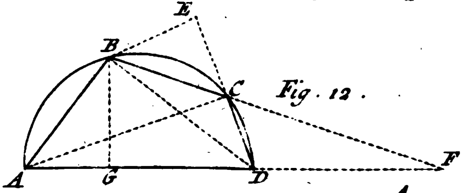


Fig. 12.

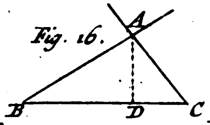


Fig. 16.

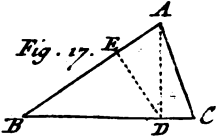
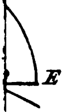


Fig. 17.



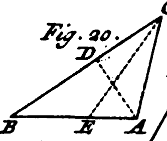
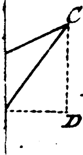


Fig. 20.



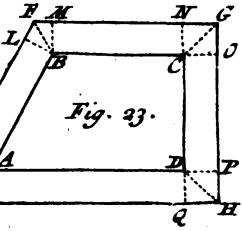


Fig. 23.

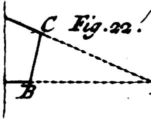


Fig. 22.

*ADE*; which make as $m$ to $d$; and the Triangle *ADE* will be $= \frac{ddxx + 2dfxy + ffyy}{dm}$. Wherefore, ſince the Area *AC*, which is the Difference of thoſe Triangles, is given, let it be $bb$, and $\frac{ddxx + 2dfxy + ffyy - dnyy}{dm}$ will be $= bb$. And ſo you have two Æquations, from [or by] the Reduction whereof all is determin'd. *Viz.* The Æquation above gives $\frac{ra - qy}{p} = x$, and by writing $\frac{ra - qy}{p}$ for $x$ in the laſt, there comes out

$$\frac{drraa - 2dqray + dqqyy}{ppm} + \frac{2afry - 2fqyy}{pm} + \frac{ffyy - dnyy}{dm} =$$

$bb$; and the Terms being abbreviated by writing $s$ for the given Quantity $\frac{dqq}{pp} - \frac{2fq}{p} + \frac{ff}{d} - n$, and $-st$ for the given $-\frac{adqr}{pp} + \frac{afr}{p}$, and $stv$ for the given $bbm - \frac{drraa}{pp}$, there ariſes $yy = 2ty + tv$, or $y = t + \sqrt{tt + tv}$.

## PROBLEM IX.

*To ſurround the Fiſh-Pond ABCD with a Walk ABCDEFGH of a given Area, and of the ſame Breadth every where.* [Vide Figure 23.]

LET the Breadth of the Walk be $x$, and its Area $aa$. And, letting fall the Perpendiculars *AK*, *BL*, *BM*, *CN*, *CO*, *DP*, *DQ*, *AI*, from the Points *A*, *B*, *C*, *D*, to the Lines *EF*, *FG*, *GH*, and *HE*, to divide the Walk into the four Trapezia *IK*, *LM*, *NO*, *PQ*, and into the four Parallelograms *AL*, *BN*, *CP*, *DI*, of the Latitude $x$, and of the ſame Length with the Sides of the given Trapezium. Let therefore the Sum of the Sides $(AB + BC + CD + DA) = b$, and the Sum of the Parallelograms will be $= bx$.

Moreover, having drawn $AE$, $BF$, $CG$, $DH$; ſince $AI$ is $=AK$, the Angle $AEI$ will be $=$ Angle $AEK$ $=\frac{1}{2}IEK$, or $\frac{1}{2}DAB$. Therefore the Angle $AEI$ is given, and conſequently the Ratio of $AI$ to $IE$, which make as $d$ to $e$, and $IE$ will be $=\frac{ex}{d}$. Multiply this into $\frac{1}{2}AI$, or $\frac{1}{2}x$, and the Area of the Triangle $AEI$ will be $=\frac{exx}{2d}$. But by reaſon of equal Angles and Sides, the Triangles $AEI$ and $AEK$ are equal, and conſequently the Trapezium $IK$ $(=2$ Triang. $AEI)$ $=\frac{exx}{d}$. In like manner, by putting $BL:LF::d:f$, and $CN:NG::d:g$, and $DP:DH::d:h$, (for thoſe Reaſons are alſo given from the given Angles $B$, $C$, and $D$) you'll have the Trapezium $LM$ $=\frac{fxx}{d}$, $NO=\frac{gxx}{d}$, and $PQ=\frac{hxx}{d}$. Wherefore $\frac{exx}{d}+\frac{fxx}{d}+\frac{gxx}{d}+\frac{hxx}{d}$, or $\frac{pxx}{d}$, by writing $p$ for $e+f+g+h$ will be equal to the four Trapeziums $IK+LM+NO+PQ$; and conſequently $\frac{pxx}{d}+bx$ will be equal to the whole Walk $aa$. Which Æquation, by dividing all the Terms by $\frac{p}{d}$, and extracting its Root, $x$ will become $=\frac{-db+\sqrt{bbdd+4aapd}}{2p}$. And the Breadth of the Walk being thus found, it is eaſy to deſcribe it.

## PROBLEM X.

*From the given Point C, to draw the right Line CF, which [together] with two other right Lines AE and AF given by Poſition, ſhall comprehend [or conſtitute] the Triangle AEF of a given Magnitude.* [Vide Figure 24.]

DRAW $CD$ parallel to $AE$, and $CB$ and $EG$ perpendicular to $AF$, and let $AD=a$, $CB=b$, and $AF=x$, and the Area of the Triangle $AEF=cc$, and by reaſon of the proportional Quantities $DF : AF \; (:: DC : AE) :: CB : EG$, that is, $a+x : x :: b : \frac{bx}{a+x}$ will be $\frac{bx}{a+x} = EG$. Multiply this into $\frac{1}{2}AF$, and there will come out $\frac{bxx}{2a+2x}$, the Quantity of the Area $AEF$, which is $=cc$. And ſo the Æquation being order'd [rightly] $xx$ will $= \frac{2ccx+2cca}{b}$, or $x = \frac{cc+\sqrt{c^4+2ccab}}{b}$.

After the ſame Manner a right Line may be drawn thro' a given Point, which ſhall divide any Triangle or Trapezium in a given Ratio.

## PROBLEM XI.

*To determine the Point C in the given right Line DF, from which the right Lines AC and BC drawn to two other Points A and B given by Poſition, ſhall have a given Difference.* [Vide Figure 25.]

FROM the given Points let fall the Perpendiculars $AD$ and $BF$ to the given right Line, and make $AD=a$, $BF=b$, $DF=c$, $DC=x$, and $AC$ will $=\sqrt{aa+xx}$, $FC=x-c$, and $BC=\sqrt{bb+xx-2cx+cc}$. Now let their given Difference be $d$, $AC$ being greater than $BC \times \sqrt{aa+xx}-d$ will $=\sqrt{bb+xx-2cx+cc}$. And ſquaring the Parts $aa+xx+dd-2d\sqrt{aa+xx}=bb+xx-2cx+cc$. And reducing, and for Abbreviation ſake, writing $2ee$ inſtead of the given [Quantities] $aa+dd-bb-cc$, there will come out $ee+cx=d \times \sqrt{aa+xx}$. And again, having ſquared the Parts $e^4+2ceex+ccxx=ddaa+ddxx$. And the Æquation being reduc'd $xx=\frac{2eecx+e^4-aadd}{dd-cc}$, or

$$x=\frac{eec+\sqrt{e^4dd-aad^4+aaddcc}}{dd-cc}.$$

The Problem will be reſolv'd after the ſame Way, if the Sum of the Lines $AC$ and $BC$, or the Sum of the Difference of their Squares, or the Proportion or Rectangle, or the Angle comprehended by them be given: Or alſo, if inſtead of the right Line $DC$, you make Uſe of the Circumference of a Circle, or any other Curve Line, ſo the Calculation (in this laſt Caſe eſpecially) relates to the Line that joyns the Points $A$ and $B$.

## PROBLEM XII.

*To determine the Point Z, from which if four right Lines ZA, ZB, ZC, and ZD are drawn at given Angles to four right Lines given by Poſition,* viz. *FA, EB, FC, GD, the Rectangle of two of the given Lines ZA and ZB, and the Sum of the other two ZC and ZB may be given.* [Vide Figure 26.]

FRom among the Lines chuſe one, as $FA$, given by Poſition, as alſo another, $ZA$, not given by Poſition, and which is drawn to it, from the Lengths whereof the Point $Z$ may be determin'd, and produce the other Lines given by Poſition till they meet theſe, or be produc'd farther out if there be Occaſion, as you ſee [here]. And having made $EA = x$, and $AZ = y$, by reaſon of the given Angles of the Triangles $AEH$, there will be given the Ratio of $AE$ to $AH$, which make as $p$ to $q$, and $AH$ will be $= \frac{qx}{p}$. Add $AZ$, and $ZH$ will be $= y + \frac{qx}{p}$. And thence, ſince by reaſon of the given Angles of the Triangle $HZB$, there is given the Ratio of $HZ$ to $BZ$, if that be made as $n$ to $p$ you'll have $ZB = \frac{py + qx}{n}$. Moreover, if the given $EF$ be called $a$, $AF$ will $= a - x$, and thence, if by reaſon of the given Angles of the Triangle $AFI$, $AF$ be made to $AI$ in the ſame Ratio as $p$ to $r$, $AI$ will become $= \frac{ra - rx}{p}$. Take this from $AZ$ and there will remain

$IZ=y-\frac{ra-rx}{p}$. And by reaſon of the given Angles of the Triangle $ICZ$, if you make $IZ$ to $ZC$ in the ſame Ratio as $m$ to $p$, $ZC$ will become $=\frac{py-ra+rx}{m}$. After the ſame Way, if you make $EG=b$. $AG:AK::l:s$, and $ZK:ZD::p:l$, there will be obtain'd $ZD=\frac{sb-sx-ly}{p}$.

Now, from the State of the Queſtion, if the Sum of the two [Lines] $ZC$ and $ZD$, $\frac{py-ra+rx}{m}+\frac{sb-sx-ly}{p}$ be made equal to any given one; and the Rectangle of the other two $\frac{pyy+qxy}{n}$ be made $=gg$, you'll have two Æquations for determining $x$ and $y$. By the latter there comes out $x=\frac{ngg-pyy}{qy}$, and by writing this Value of $x$ in the room of that in the former Æquation, there will come out $\frac{py-ra}{m}+\frac{rngg-rpyy}{mqy}+\frac{sb-ly}{p}-\frac{sngg-spyy}{pqy}=f$; and by Reduction $yy=\frac{apqry-bmqsy+fmpqy+ggmns-ggnpr}{ppq-ppr-mpq+mps}$; and for Abbreviation ſake, writing $2h$ for $\frac{apqr-bmqs+fmpq}{ppq-ppr-mlq+mps}$, and $kk$ for $\frac{ggmns-ggpnr}{ppq-ppr-mlq+mps}$, you'll have $yy=2hy+kk$, or $y=h\pm\sqrt{hh+kk}$. And ſince $y$ is known by means of this Æquation, the Æquation $\frac{ngg-pyy}{qy}=x$ will give $x$. Which is ſufficient to determine the Point $z$.

After the ſame Way a Point is determin'd [or may be determin'd] from which other right Lines may be drawn to more or fewer right Lines given by Poſition, ſo that the Sum, or Difference, or Rectangle of ſome of them may be given, or may be made equal to the Sum, or Difference, or Rectangle of the reſt, or that they may have any other aſſign'd Conditions.

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## PROBLEM XIII.

*To ſubtend the right Angle EAF with the right Line EF given in Magnitude, which ſhall paſs through the given Point C, [which ſhall be] æquidiſtant from the Lines that comprehend the right Angle (when they are produc'd).* [Vide Figure 27.]

Complete the Square *ABCD*, and biſect the Line *EF* in *G*. Then call *CB* or *CD*, $a$; *EG* or *FG*, $b$; and *CG*, $x$; and *CE* will $= x - b$, and $CF = x + b$. Then ſince $CFq - BCq = BFq$, *BF* will $= \sqrt{xx + 2bx + bb - aa}$. Laſtly, by reaſon of the ſimilar Triangles *CDE*, *FBC*, $CE : CD :: CF : BF$, or $x - b : a :: x + b : \sqrt{xx + 2bx + bb - aa}$. Whence $ax + ab = \overline{x - b} \times \sqrt{xx + 2bx + bb - aa}$. Each part of which Æquation being ſquar'd, and the Terms that come out being reduc'd into Order, there comes out $x^4 = \begin{matrix} 2aa \\ +2bb \end{matrix} xx \begin{matrix} +2aabb \\ -b^4 \end{matrix}$. And extracting the Root as in Quadratick Æquations, there comes out $xx = aa + bb + \sqrt{a^4 + 4aabb}$; and conſequently $x = \sqrt{aa + bb + \sqrt{a^4 + 4aabb}}$. And *CG* being thus found, gives *CE* or *CF*, which, by determining the Point *E* or *F*, ſatisfies the Problem.

*The ſame otherwiſe:*

Let *CE* be $= x$, $CD = a$, and $EF = b$; and *CF* will be $= x + b$, and $BF = \sqrt{xx + bb + 2bx - aa}$. And then ſince $CE : CD :: CF : BF$, or $x : a :: x + b : \sqrt{xx + 2bx + bb - aa}$, $ax + ab$ will be $= x \times \sqrt{xx + 2bx + bb - aa}$. The Parts of this Æquation being ſquar'd, and the Terms reduc'd into Order, there will come out $x^4 + 2bx^3 \begin{matrix} +bb \\ -2aa \end{matrix} xx - 2aabx - aabb = 0$, a Biquadratick Æquation, the Inveſtigation of the Root of which is more difficult than in the former Caſe. But it may be thus inveſtigated; put $x^4 + 2bx^3 \begin{matrix}+bb\\-2aa\end{matrix} xx - 2aabx + a^4 = aabb + a^4$, and extracting the Root on both Sides $xx + bx - aa = \pm a\sqrt{aa+bb}$.

Hence I have an Opportunity of giving a Rule for the Election of Terms for the Calculus, *viz.* when there happens to be ſuch an Affinity or Similitude of the Relation of two Terms to the other Terms of the Queſtion, that you ſhould be oblig'd in making Uſe of either of them to bring out Æquations exactly alike; or that both, if they are made Uſe of together, ſhall bring out the ſame Dimenſions and the ſame Form in the final Æquation, (only excepting perhaps the Signs + and —, which will be eaſily [and readily] ſeen) then it will be the beſt Way to make Uſe of neither of them, but in their room to chuſe ſome third, which ſhall bear a like Relation to both, as ſuppoſe the half Sum, or half Difference, or perhaps a mean Proportional, or any other Quantity related to both indifferently and without a like [before made Uſe of]. Thus, in the precedent Problem, when I ſee the Line *EF* alike related to both *AB* and *AD*, (which will be evident if you alſo draw *EF* in the Angle *BAH*) and therefore I can by no Reaſon be perſwaded why *ED* ſhould be rather made Uſe of than *BF*, or *AE* rather than *AF*, or *CE* rather than *CF* for the Quantity ſought: Wherefore, in the room of the Points *C* and *F*, from whence this Ambiguity comes, (in the former Solution) I made Uſe of the intermediate [Point] *G*, which has [or bears] a like Relation to both the Lines *AB* and *AD*. Then from this Point *G*, I did not let fall a Perpendicular to *AF* for finding the Quantity ſought, becauſe I might by the ſame Ratio have let one fall to *AD*. And therefore I let it fall upon neither *CB* nor *CD*, but propos'd *CG* for the Quantity ſought, which does not admit of a like; and ſo I obtain'd a Biquadratick Æquation without the odd Terms.

I might alſo (taking Notice that the Point *G* lies in the Periphery of a Circle deſcrib'd from the Center *A*, by the Radius *EG*) have let fall the Perpendicular *GK* upon the Diagonal *AC*, and have ſought *AK* or *CK*, (as which bear alſo a like Relation to both *AB* and *AD*) and ſo I ſhould have fall'n upon a Quadratick Æquation, *viz.* $yy = \frac{1}{2}ey + \frac{1}{2}bb$, making $AK = y$, $AC = e$, and $AG = b$. And *AK* being ſo found, there muſt have been erected the Perpendicular $KG$ meeting the aforeſaid Circle in $G$, thro' which $CF$ would paſs.

Taking particular Notice of this Rule in *Prob.* 5 *and* 6, where the Sides $BC$ and $AC$ were to be determin'd, I rather ſought the Semi-difference than either of them. But the Uſefulneſs of this Rule will be more evident from the following Problem.

## PROBLEM XIV.

*So to inſcribe the right Line DC of a given Length in the given Conick Section DAC, that it may paſs through the Point G given by Poſition.* [Vide Figure 28.]

LET $AF$ be the Axis of the Curve, and from the Points $D$, $G$, and $C$ let fall to it the Perpendiculars $DH$, $GE$, and $CB$. Now to determine the Poſition of the right Line $DC$, it may be propos'd to find out the Point $D$ or $C$; but ſince theſe are related, and ſo alike, that there would be the like Operation in determining either of them, whether I were to ſeek $CG$, $CB$, or $AB$; or their likes, $DG$, $DH$, or $AH$; therefore I look after a third Point, that regards $D$ and $C$ alike, and at the ſame time determines them. And I ſee $F$ to be ſuch a Point.

Now let $AE=a$, $EG=b$, $DC=c$, and $EF=z$; and beſides, ſince the Relation between $AB$ and $BC$ in the Æquation, I ſuppoſe, given for determining the Conick Section, let $AB=x$, $BC=y$, and $FB$ will be $=x-a+z$. And becauſe $GE:EF::CB:FB$, $FB$ will again be $=\frac{yz}{b}$. Therefore, $x-a+z=\frac{yz}{b}$. Theſe Things being thus laid down, take away $x$, by the Equation that denotes [or expreſſes] the Curve. As if the Curve be a Parabola expreſs'd by the Æquation $rx=yy$, write $\frac{yy}{r}$ for $x$; and there will ariſe $\frac{yy}{r}-a+z=\frac{yz}{b}$, and extracting the Root $y=\frac{rz}{2b}\pm\sqrt{\frac{rrzz}{4bb}+ar-rz}$. Whence it is evident, that $\sqrt{\frac{rrzz}{bb}+4ar-4rz}$ is the Difference of the double Value of $y$, that is, of the Lines $+BC$ and $-DH$, and conſequently (having let fall $DK$ perpendicular upon $CB$) that Difference is equal to $CK$. But $FG : GE :: DC : CK$, that is, $\sqrt{bb+zz} : b :: c : \sqrt{\frac{rrzz}{bb}+4ar-4rz}$. And by multiplying the Squares of the Means, and alſo the Squares of the Extreams into one another, and ordering the Products, there will ariſe $z^4 =$

$$\frac{4bbrz^3 \begin{array}{l}-4abbr\\-bbrr\end{array} zz + 4b^4rz \begin{array}{l}-4ab^4r\\+b^4cc\end{array}}{rr},$$

an Æquation of four Dimenſions, which would have riſen to one of eight Dimenſions if I had ſought either $CG$, or $CB$, or $AB$.

## PROBLEM XV.

*To multiply or divide a given Angle by a given Number.* [Vide Figure 29.]

IN any Angle $FAG$ inſcribe the Lines $AB$, $BC$, $CD$, $DE$, *&c.* of any the ſame Length, and the Triangles $ABC$, $BCD$, $CDE$, $DEF$, *&c.* will be *Iſoſceles*, and conſequently by the 32. 1. *Eucl.* the Angle $CBD$ will be $=$ Angle $A+ACB = 2$ Angle $A$, and the Angle $DCE =$ Angle $A+ADC = 3$ Angle $A$, and the Angle $EDF = A+AED = 4$ Angle $A$, and the Angle $FEG = 5$ Angle $A$, and ſo onwards. Now, making $AB$, $BC$, $CD$, *&c.* the Radii of equal Circles, the Perpendiculars $BK$, $CL$, $DM$, *&c.* let fall upon $AC$, $BD$, $CE$, *&c.* will be the Sines of thoſe Angles, and $AK$, $BL$, $CM$, $DN$, *&c.* will be their Sines Complement to a right one; or making $AB$ the Diameter, the Lines $AK$, $BL$, $CM$, *&c.* will be Chords. Let therefore $AB = 2r$, and $AK = x$, then work thus:

$$AB : AK :: AC : AL.$$

$$2r : x :: 2x : \frac{xx}{r}.$$

And

$$\text{And } \left\{\begin{array}{l} AL - AB \\ \frac{xx}{r} - 2r \end{array}\right\} = BL, \text{ the Duplication.}$$

$$AB : AK :: AD\ (2AL - AB) : AM.$$

$$2r : x :: \frac{2xx}{r} - 2r : \frac{x^3}{rr} - x.$$

$$\text{And } \left\{\begin{array}{l} AM - AC \\ \frac{x^3}{rr} - 3x \end{array}\right\} = CM, \text{ the Triplication.}$$

$$AB : AK :: AE\ (2AM - AC) : AN.$$

$$2r : x :: \frac{2x^3}{rr} - 4x : \frac{x^4}{r^3} - \frac{2xx}{r}.$$

$$\text{And } \left\{\begin{array}{l} AN - AD \\ \frac{x^4}{r^3} - \frac{4xx}{r} + 2r \end{array}\right\} = DN, \text{ the Quadruplication.}$$

$$AB : AK :: AF\ (2AN - AD) : AO.$$

$$2r : x :: \frac{2x^4}{r^3} - \frac{6xx}{r} + 2r : \frac{x^5}{r^4} - \frac{3x^3}{rr} + x.$$

$$\text{And } \left\{\begin{array}{l} AO - AE \\ \frac{x^5}{r^4} - \frac{5x^3}{rr} + 5x \end{array}\right\} = EO, \text{ the Quintuplication.}$$

And ſo onwards. Now if you would divide an Angle into any Number of Parts, put $q$ for $BL$, $CM$, $DN$, &c. and you'll have $xx - 2rr = qr$ for the Biſection; $xxx - 3rrx = qr^2$ for the Triſection; $xxxx - 4rrxx + 2r^4 = qr^3$ for the Quadriſection; $xxxxx - 5r^2x^3 + 5r^4x = qr^4$ for the Quinquiſection, &c.

## PROBLEM XVI.

### *To determine the Poſition of a Comet's Courſe* [*or Way*] *that moves uniformly in a right Line,* [*as*] *BD, from three Obſervations.* [Vide Figure 30.]

SUppoſe $A$ to be the Eye of the Spectator, $B$ the Place of the Comet in the firſt Obſervation, $C$ in the ſecond, and $D$ in the third; the Inclination of the Line $BD$ to the Line $AB$ is to be found. From the Obſervations therefore there are given the Angles $BAC$, $BAD$; and conſequently if $BH$ be drawn perpendicular to $AB$, and meeting $AC$ and $AD$ in $E$ and $F$, aſſuming any how $AB$, there will be given $BE$ and $BF$, *viz.* the Tangents of the Angles in reſpect of the Radius $AB$. Make therefore $AB=a$, $BE=b$, and $BF=c$. Moreover, from the given Intervals [or Diſtances] of the Obſervations, there will be given the Ratio of $BC$ to $BD$, which, if it be made as $b$ to $e$, and $DG$ be drawn parallel to $AC$, ſince $BE$ is to $BG$ in the ſame Ratio, and $BE$ was call'd $b$, $BG$ will be $=e$, and conſequently $GF=e-c$. Moreover, if you let fall $DH$ perpendicular to $BG$, by reaſon of the Triangles $ABF$ and $DHF$ being like, and alike divided by the Lines $AE$ and $DG$, $FE$ will be $:AB::FG:HD$, that is, $c-b:a::e-c:\frac{ae-ac}{c-b}=HD$. Moreover, $FE$ will be $:FB::FG:FH$, that is, $c-b:c::e-c:\frac{ce-cc}{c-b}=FH$; to which add $BF$, or $c$, and $BH$ will be $=\frac{ce-cb}{c-b}$. Wherefore $\frac{ce-cb}{c-b}$ is to $\frac{ae-ac}{c-b}$ (or $ce-cb$ to $ae-ac$, or $\frac{ce-cb}{e-c}$ to $a$) as $BH$ to $HD$; that is, as the Tangent of the Angle $HDB$, or $ABK$ to the Radius. Wherefore, ſince $a$ is ſuppos'd to be the Radius, $\frac{ce-cb}{e-c}$ will be the Tangent of the Angle $ABK$, and therefore by reſolving [them into an Analogy] 'twill be as $e-c$ to $e-b$, (or $GF$ to $GE$) ſo $c$ (or the Tangent of the Angle $BAF$) to the Tangent of the Angle $ABK$.

Say therefore, as the Time between the firſt and ſecond Obſervation to the Time between the firſt and third, ſo the Tangent of the Angle $BAE$ to a fourth Proportional. Then as the Difference between that fourth Proportional and the Tangent of the Angle $BAF$, to the Difference between the ſame fourth Proportional and the Tangent of the Angle $BAE$, ſo the Tangent of the Angle $BAF$ to the Tangent of the Angle $ABK$.

## PROBLEM XVII.

*Rays [of Light] from any ſhining or lucid Point diverging to a refracting Spherical Surface, to find the Concourſe of each of the refracted Rays with the Ax of the Sphere paſſing thro' that lucid Point.* [Vide Figure 31.]

LET $A$ be that lucid Point, and $BV$ the Sphere, the Axis whereof is $AD$, the Center $C$, and the Vertex $V$; and let $AB$ be the incident Ray, and $BD$ the refracted Ray; and having let fall to thoſe Rays the Perpendiculars $CE$ and $CF$, as alſo $BG$ perpendicular to $AD$, and having drawn $BC$, make $AC=a$, $VC$ or $BC=r$, $CG=x$, and $CD=z$, and $AG$ will be $=a-x$, $BG=\sqrt{rr-xx}$, $AB=\sqrt{aa-2ax+rr}$; and by reaſon of the ſimilar Triangles $ABG$ and $ACE$, $CE$ will $=\dfrac{a\sqrt{rr-xx}}{\sqrt{aa-2ax+rr}}$. Alſo $GD=z+x$, $BD=\sqrt{zz+2zx+rr}$; and by reaſon of the ſimilar Triangles $DBG$ and $DCF$, $CF=\dfrac{z\sqrt{rr-xx}}{\sqrt{zz+2zx+rr}}$. Beſides, ſince the Ratio of the Sines of Incidence and Refraction, and conſequently of $CE$ to $CF$, is given, ſuppoſe that Ratio to be as $a$ to $f$, and $\dfrac{fa\sqrt{rr-xx}}{\sqrt{aa-2ax+rr}}$ will be $=\dfrac{az\sqrt{rr-xx}}{\sqrt{zz+2zx+rr}}$; and multiplying croſs-ways, and dividing by $a\sqrt{rr-xx}$, $f\sqrt{zz+2zx+rr}$ will be $=z\sqrt{aa-2xa+rr}$, and by ſquaring and reducing the Terms into Order, $zz = \frac{2ffxz+ffrr}{aa-2ax+rr-ff}$. Then for the given $\frac{ff}{a}$ write $p$, and $q$ for the given $a+\frac{rr}{a}-p$, and $zz$ will be $=\frac{2pxz+prr}{q-2x}$, and $z=\frac{px+\sqrt{ppxx-2prrx+pqrr}}{q-2x}$. Therefore $z$ is found; that is, the Length of $CD$, and conſequently the Point ſought $D$, where the refracted Ray $BD$ meets with the Axis. Q. E. F.

Here I made the incident Rays to diverge, and fall upon a thicker Medium; but changing what is requiſite to be changed, the Problem may be as eaſily reſolved when the Rays converge, or fall from a thicker Medium into a thinner one.

## PROBLEM XVIII.

### *If a Cone be cut by any Plane, to find the Figure of the Section.* [Vide Figure 32.]

LET $ABC$ be a Cone ſtanding on a circular Baſe $BC$, and $IEM$ its Section ſought; and let $KILM$ be any other Section parallel to the Baſe, and meeting the former Section in $HI$; and $ABC$ a third Section, perpendicularly biſecting the two former in $EH$ and $KL$, and the Cone in the Triangle $ABC$, and producing $EH$ till it meet $AK$ in $D$; and having drawn $EF$ and $DG$ parallel to $KL$, and meeting $AB$ and $AC$ in $F$ and $G$, call $EF=a$, $DG=b$, $ED=c$, $EH=x$, and $HI=y$; and by reaſon of the ſimilar Triangles $EHL$, $EDG$, $ED$ will be $: DG :: EH : HL = \frac{bx}{c}$. Then by reaſon of the ſimilar Triangles $DEF$, $DHK$, $DE$ will be $: EF :: DH :$ ($c-x$ in the firſt Figure, and $c+x$ in the ſecond Figure) $HK = \frac{ac\mp ax}{c}$. [*Vide Figure* 33.] Laſtly, ſince the Section $KIL$ is parallel to the Baſe, and conſequently circular,

$HK \times HL$ will be $= HIq$, that is, $\frac{ab}{c}x \mp \frac{ab}{cc}xx = yy$, an Æquation which expreſſes the Relation between $EH$ $(x)$ and $HI$ $(y)$, that is, between the Axis and the Ordinate of the Section $EIM$; which Æquation, ſince it expreſſes an Ellipſe in the firſt Figure, and an Hyperbola in the ſecond Figure, it is evident, that that Section will be Ellipſical or Hyperbolical.

Now if $ED$ no where meets $AK$, being parallel to it, then $HK$ will be $= EF$ $(a)$, and thence $\frac{ab}{c}x$ $(HK \times HL)$ $= yy$, an Æquation expreſſing a Parabola.

## PROBLEM XIX.

*If the right Line XY be turn'd about the Axis AB, at the Diſtance CD, with a given Inclination to the Plane DCB, and the Solid PQRUTS, generated by that Circumrotation, be cut by any Plane [as] INQLK, to find the Figure of the Section.* [Vide Figure 34.]

LET $BHQ$, or $GHO$ be the Inclination of the Axis $AB$ to the Plane of the Section; and let $L$ be any Concourſe of the right Line $XY$ with that Plane. Draw $DF$ parallel to $AB$, and let fall the Perpendiculars $LG$, $LF$, $LM$, to $AB$, $DF$, and $HO$, and join $FG$ and $MG$. And having call'd $CD = a$, $CH = b$, $HM = x$, and $ML = y$, by reaſon of the given Angle $GHO$, making $MH : HG :: d : e$, $\frac{ex}{d}$ will $= GH$, and $b + \frac{ex}{d} =$ to $GC$ or $FD$. Moreover, by reaſon of the given Angle $LDF$ (*viz.* the Inclination of the right Line $XY$ to the Plane $GCDF$) putting $FD : FL :: g : h$, $\frac{hb}{g} + \frac{hex}{dg} = FL$, to whoſe Square add $FGq$ $(DCq$, or $aa)$ and there will come out $GLq = aa + \frac{hhbb}{gg} + \frac{2hhbex}{dgg} + \frac{hheexx}{ddgg}$. Hence ſubtract $MGq$ $(HMq - HGq$, or $xx - \frac{ee}{dd}xx)$ and there will remain $\frac{aagg+hhbb}{gg}+\frac{2hhbe}{dgg}x+\frac{hhee-ddgg+eegg}{ddgg}$ $\times xx$ $(=MLq)=yy$: an Æquation that expreſſes the Relation between $x$ and $y$, that is, between $HM$ the Axis of the Section, and $ML$ its Ordinate. And therefore, ſince in this Æquation $x$ and $y$ aſcend only to two Dimenſions, it is evident, that the Figure $INQLK$ is a Conick Section. As for Example, if the Angle $MHG$ is greater than the Angle $LDF$, this Figure will be an Ellipſe; but if leſs, an Hyperbola; and if equal, either a Parabola, or (the Points $C$ and $H$ moreover coinciding) a Parallelogram.

## PROBLEM XX.

*If you erect $AD$ of a given Length perpendicular to $AF$, and $ED$, one Leg of a Square $DEF$, paſs continually thro' the Point $D$, while the other Leg $EF$ equal to $AD$ ſlide upon $AF$, to find the Curve $HIC$, which the Leg $EF$ deſcribes by its middle Point $C$.* [Vide Figure 35.]

LET $EC$ or $CF=a$, the Perpendicular $CB=y$, $AB=x$, and $BF$ $(\sqrt{aa-yy}):BC+CF$ $(y+a)::$ $EF$ $(2a):EG+GF=(AG+GF)$ or $AF$. Wherefore $\frac{2ay+2aa}{\sqrt{aa-yy}}$ $(=AF=AB+BF)$ $=x+\sqrt{aa-yy}$. Now, by multiplying by $\sqrt{aa-yy}$ there is made $2ay+2aa=aa-yy+x\sqrt{aa-yy}$, or $2ay+aa+yy=x\times\sqrt{aa-yy}$, and by ſquaring the Parts, and dividing by $\sqrt{a+y}$, and ordering them, there comes out $y^3+3ayy\begin{array}{l}+3aa\\+xx\end{array}y\begin{array}{l}+a^3\\-axx\end{array}=0$.

*The ſame otherwiſe.* [Vide Figure 36.]

On $BC$ take at each End $BI$, and $CK$ equal to $CF$, and draw $KF$, $HI$, $HC$, and $DF$; whereof $HC$ and $DF$ meet $AF$, and $IK$ in $M$ and $N$, and upon $HC$ let fall the Perpendicular $IL$; and the Angle $K$ will be $= \frac{1}{2} BCF = \frac{1}{2} EGF = GFD = AMH = MHI = CIL$; and conſequently the right-angled Triangles $KBF$, $FBN$, $HLI$, and $ILC$ will be ſimilar. Make therefore $FC = a$, $HI = x$, and $IC = y$; and $BN$ $(2a - y)$ will be $: BK$ $(y)$ $:: LC : LH :: CIq$ $(yy)$ $: HIq$ $(xx)$, and conſequently $2axx - yxx = y^3$. From which Æquation it is eaſily inferr'd, that this Curve is the Ciſſoid of the Antients, belonging to a Circle, whoſe Center is $A$, and its Radius $AH$.

## PROBLEM XXI.

*If a right Line ED of a given Length ſubtending the given Angle EAD, be ſo moved, that its Ends D and E always touch the Sides AD and AE of that Angle; let it be propos'd to determine the Curve FCG, which any given Point C in that right Line ED deſcribes.* [Vide Figure 37.]

FROM the given Point $C$ draw $CB$ parallel to $EA$; and make $AB = x$, $BC = y$, $CE = a$, and $CD = b$, and by reaſon of the ſimilar Triangles $DCB$, $DEA$, $EC$ will be $: AB :: CD : BD$; that is, $a : x :: b : BD = \frac{bx}{a}$. Beſides, having let fall the Perpendicular $CH$, by reaſon of the given Angle $DAE$, or $DBC$, and conſequently of the given Ratio of the Sides of the right-angled Triangle $BCH$, you'll have $a : e :: BC : BH$, and $BH$ will be $= \frac{ey}{a}$. Take away this from $BD$, and there will remain $HD = \frac{bx - ey}{a}$. Now in the Triangle $BCH$, becauſe of the right Angle $BHC$, $BCq - BHq = CHq$; that is, $yy - \frac{eeyy}{aa} = CHq$. In like manner, in the Triangle $CDH$, becauſe of the right Angle $CDH$, $CDq - CHq$ is $= HDq$; that is, $bb - yy + \frac{eeyy}{aa}$ $\left(= HDq = \frac{bx - ey}{a} q.\right)$

$= bbxx$

Fig. 25.

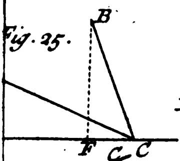


Fig. 26.

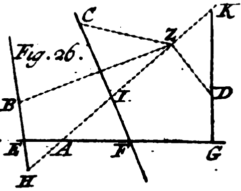


Fig. 28.

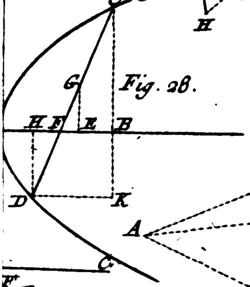
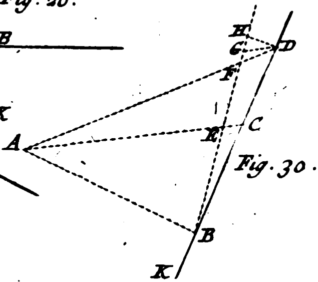
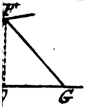


Fig. 30.





g. 31.

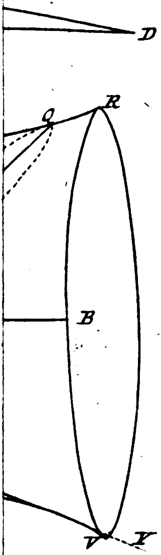


Fig. 32.

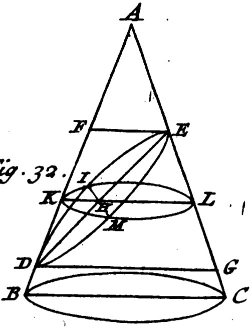
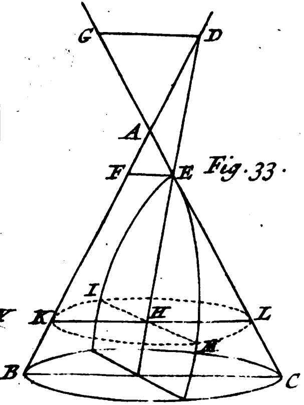


Fig. 33.



$= \frac{bbxx - 2bexy + eeyy}{aa}$; and by Reduction $yy = \frac{2be}{aa} xy + \frac{aabb - bbxx}{aa}$. Where, ſince the unknown Quantities are of two Dimenſions, it is evident that the Curve is a Conick Section. Then extracting the Root, you'll have $y = \frac{bex \mp b\sqrt{eexx - aaxx + a^4}}{aa}$. Where, in the Radical Term, the Coefficient of $xx$ is $ee - aa$. But it was $a : e :: BC : BH$; and $BC$ is neceſſarily a greater Line than $BH$, *viz.* the Hypothenuſe of a right-angled Triangle is greater than the Side of it; therefore $a$ is greater than $e$, and $ee - aa$ is a negative Quantity, and conſequently the Curve will be an Ellipſis.

## PROBLEM XXII.

*If the Ruler EBD, forming a right Angle, be ſo moved, that one Leg of it, EB, continually ſubtends the right Angle EAB, while the End of the other Leg, BD, deſcribes ſome Curve Line, as FD; to find that Line FD, which the Point D deſcribes.* [Vide Figure 38.]

FROM the Point $D$ let fall the Perpendicular $DC$ to the Side $AC$; and making $AC = x$, and $DC = y$, and $EB = a$, and $BD = b$. In the Triangle $BDC$, by reaſon of the right Angle at $C$, $BCq$ is $= BDq - DCq = bb - yy$. Therefore $BC = \sqrt{bb - yy}$; and $AB = x - \sqrt{bb - yy}$. Beſides, by reaſon of the ſimilar Triangles $BEA$, $DBC$, $BD : DC :: EB : AB$; that is, $b : y :: a : x - \sqrt{bb - yy}$; therefore $bx - b\sqrt{bb - yy} = ay$, or $bx - ay = b\sqrt{bb - yy}$. And the Parts being ſquar'd and duly reduc'd $yy = \frac{2abxy + b^4 - bbxx}{aa + bb}$, and extracting the Root $y = \frac{abx \pm bb\sqrt{aa + bb - xx}}{aa + bb}$. Whence it is again evident, that the Curve is an Ellipſe.

This is ſo where the Angles $EBD$ and $EAB$ are right; but if thoſe Angles are of any other Magnitude, as long as they are equal, you may proceed thus: [*Vide Figure* 39.] Let fall $DC$ perpendicular to $AC$ as before, and draw $DH$, making the Angle $DHA$ equal to the Angle $HAE$, ſuppoſe Obtuſe, and calling $EB = a$, $BD = b$, $AH = x$, and $HD = y$; by reaſon of the ſimilar Triangles $EAB$, $BHD$, $BD$ will be : $DH :: EB : AB$; that is, $b : y :: a : AB = \frac{ay}{b}$. Take this from $AH$ and there will remain $BH = x - \frac{ay}{b}$. Beſides, in the Triangle $DHC$, by reaſon of all the Angles given, and conſequently the Ratio of the Sides given, aſſume $DH$ to $HC$ in any given Ratio, ſuppoſe as $b$ to $e$; and ſince $DH$ is $y$, $HC$ will be $\frac{ey}{b}$, and $HB \times HC$ will $= \frac{exy}{b} - \frac{aeyy}{bb}$. Laſtly, by the 12, 2 *Elem.* in the Triangle $BHD$, $BDq$ is $= BHq + DHq + 2BH \times HC$; that is, $bb = xx - \frac{2axy}{b} + \frac{aayy}{bb} + yy + \frac{2exy}{b} - \frac{2aeyy}{bb}$. and extracting the Root $x = \frac{ay - ey \pm \sqrt{eeyy - bbyy + bbbb}}{b}$. Where, when $b$ is greater than $e$, that is, when $ee - bb$ is a negative Quantity, it is again evident, that the Curve is an Ellipſe.

## PROBLEM XXIII.

*Having the Sides and Baſe of any right-lined Triangle given, to find the Segments of the Baſe, the Perpendicular, the Area, and the Angles.* [Vide Figure 40.]

LET there be given the Sides $AC$, $BC$, and the Baſe $AB$ of the Triangle $ABC$. Biſect $AB$ in $I$, and take on it (being produc'd on both Sides) $AF$ and $AE$ equal to $AC$, and $BG$ and $BH$ equal to $BC$. Join $CE$, $CF$; and from $C$ to the Baſe let fall the Perpendicular $CD$. And $ACq - BCq$ will be $= ADq + CDq - CDq - BDq = ADq$

$= ADq - BDq = \overline{AD + BD} \times \overline{AD - BD} = AB \times 2DI$. Therefore $\frac{ACq - BCq}{2AB} = DI$. And $2AB : AC + BC :: AC - BC : DI$. Which is a Theorem for determining the Segments of the Baſe.

From $IE$, that is, from $AC - \frac{1}{2}AB$, take away $DI$, and there will remain $DE = \frac{BCq - ACq + 2AC \times AB - ABq}{2AB}$, that is, $= \frac{\overline{BC + AC - AB} \times \overline{BC - AC + AB}}{2AB}$, or $= \frac{HE \times EG}{2AB}$. Take away $DE$ from $FE$, or $2AC$, and there will remain $FD = \frac{ACq + 2AC \times AB + ABq - BCq}{2AB}$; that is, $= \frac{\overline{AC + AB + BC} \times \overline{AC + AB - BC}}{2AB}$, or $= \frac{FG \times FH}{2AB}$. And ſince $CD$ is a mean Proportional between $DE$ and $DF$, and $CE$ a mean Proportional between $DE$ and $EF$, and $CF$ a mean Proportional between $DF$ and $EF$, $CD$ will be $= \frac{\sqrt{FG \times FH \times HE \times EG}}{2AB}$, $CE = \sqrt{\frac{AC \times HE \times EG}{AB}}$, and $CF = \sqrt{\frac{AC \times FG \times FH}{AB}}$. Multiply $CD$ into $\frac{1}{2}AB$, and you'll have the Area $= \frac{1}{4}\sqrt{FG \times FH \times HE \times EG}$. But for determining the Angle $A$, there come out ſeveral Theorems:

1. As $2AB \times AC : HE \times EG$ $(:: AC : DE)$ $::$ Radius : verſed Sine of the Angle $A$.
2. $2AB \times AC : \sqrt{FG \times FH}$ $(:: AC : FD)$ $::$ Radius : verſed Coſine of $A$.
3. $2\overline{AB \times AC} : \sqrt{FG \times FH \times HE \times EG}$ $(:: AC : CD)$ $::$ Radius : Sine of $A$.
4. $\sqrt{FG \times FH} : \sqrt{HE \times EG}$ $(:: CF : CE)$ $::$ Radius : Tangent of $\frac{1}{2}A$.
5. $\sqrt{HE \times EG} : \sqrt{FG \times FH}$ $(:: CE : FC)$ $::$ Radius : Cotangent of $\frac{1}{2}A$.

6. $2\sqrt{AB \times AC} : \sqrt{HE \times EG}$ ($: : FE : CE$) : : Radius : Sine of $\frac{1}{2}A$.

7. $2\sqrt{AB \times AC} : \sqrt{FG \times FH}$ ($: : FE : FC$) : : Radius : Coſine of $\frac{1}{2}A$.

## PROBLEM XXIV.

*In the given Angle PAB having any how drawn the right Lines, BD, PD, in a given Ratio, on this Condition, that BD ſhall be parallel to AP, and PD terminated at the given Point P in the right Line AP; to find the Locus of the Point D.* [Vide Figure 41.]

DRAW $CD$ parallel to $AB$, and $DE$ perpendicular to $AP$; and make $AP = a$, $CP = x$, and $CDy$, and let $BD$ be to $PD$ in the ſame Ratio as $d$ to $e$, and $AC$ or $BD$ will be $= a - x$, and $PD = \frac{ea - ex}{d}$. Moreover, by reaſon of the given Angle $DCE$, let the Ratio of $CD$ to $CE$ be as $d$ to $f$, and $CE$ will be $= \frac{fy}{d}$, and $EP = x - \frac{fy}{d}$. But by reaſon of the Angles at $E$ being right ones, $CDq - CEq$ will be $(= EDq) = PDq - EPq$; that is, $yy - \frac{ffyy}{dd} = \frac{eeaa - 2eeax + eexx}{dd} - xx + \frac{2fxy}{d} - \frac{ffyy}{dd}$, and blotting out on each Side $-\frac{ffyy}{dd}$, and the Terms being rightly diſpos'd, $yy = \frac{2fxy}{d} + \frac{eeaa - 2eeax + eexx - ddxx}{dd}$, and extracting the Root

$$y = \frac{fx}{d} \mp \sqrt{\frac{eeaa - 2eeax \begin{array}{l} +ce \\ -dd \\ +ff \end{array} xx}{dd}}.$$

Where, ſince $x$ and $y$ in the laſt Æquation aſcends only to two Dimenſions, the Place of the Point $D$ will be a Conick Section, and that either an Hyperbola, Parabola, or Ellipſe, as $ee - dd + ff$, (the Co-efficient of $xx$ in the laſt Æquation) is greater, equal to, or leſs than nothing.

## PROBLEM XXV.

*The two right Lines VE and VC being given in Poſition, and cut any how in C and E by another right Line, PE turning about the Pole, P given alſo in Poſition; if the intercepted Line CE be divided into the Parts CD, DE that have a given Ratio to one another, it is propos'd to find the Place of the Point D.* [Vide Figure 42.]

DRAW $VP$, and parallel to it $DA$, and $EB$ meeting $VC$ in $A$ and $B$. Make $VP = a$, $VA = x$, and $AD$ $y$, and ſince the Ratio of $CD$ to $DE$ is given, or conſely of $CD$ to $CE$, that is, the Ratio of $DA$ to $EB$, let e as $d$ to $e$, and $EB$ will be $= \frac{ey}{d}$. Beſides, ſince the gles $EVB$, $EVP$ are given, and conſequently the Ratio $EB$ to $VB$, let that Ratio be as $e$ to $f$, and $VB$ will be $=$

Laſtly, by reaſon of the ſimilar Triangles $CEB$, $CDA$, $V$, $EB : CB :: DA : CA :: VP : VC$, and by Compoion $EB + VP : CB + VC :: DA + VP : CA + VC$; that $\frac{ey}{d} + a : \frac{fy}{d} :: y + a : x$, and multiplying together the ans and Extremes $eyx + dax = fyy + fay$. Where, ſince the indefinite Quantities $x$ and $y$ aſcend only wo Dimenſions, it follows, that the Curve $VD$, in which Point $D$ is always found, is a Conick Section, and that Hyperbola, becauſe one of the indefinite Quantities, *viz.* only of one Dimenſion, and the Term $exy$ is multid by the other indefinite one $y$.

## PROBLEM XXVI.

*If two right Lines, AC and AB, in any given Ratio, are drawn from the two Points A and B given in Poſition, to a third Point C, to find the Place of C, the Point of Concourſe.* [Vide Figure 43.]

JOIN *AB*, and let fall to it the Perpendicular *CD*; and making $AB=a$, $AD=x$, $DC=y$, *AC* will be $=\sqrt{xx+yy}$, $BD=x-a$, and *BC* $(=\sqrt{BDq+DCq})$ $=\sqrt{xx-2ax+aa+yy}$. Now ſince there is given the Ratio of *AC* to *BC*, let that be as *d* to *e*; and the Means and Extremes being multiply'd together, you'll have $e\sqrt{xx+yy}=d\sqrt{xx-2ax+aa+yy}$, and by Reduction $\sqrt{\frac{ddaa-2ddax}{ee-dd}-xx}=y$. Where, ſince *x* is Negative, and affected only by Unity, and alſo the Angle *ADC* a right one, it is evident, that the Curve in which the Point *C* is plac'd is a Circle, *viz.* in the right Line *AB* take the Points *E* and *F*, ſo that $d:e::AE:BE::AF:BF$, and *EF* will be the Diameter of this Circle.

And hence from the Converſe this Theorem comes out, that in the Diameter of any Circle *EF* being produc'd, having given any how the two Points *A* and *B* on this Condition, that $AE:AF::BE:BF$, and having drawn from theſe Points the two right Lines *AC* and *BC*, meeting the Circumference in any Point *C*; *AC* will be to *BC* in the given Ratio of *AE* to *BE*.

## PROBLEM XXVII.

*To find the Point D, from which three right Lines DA, DB, DC, let fall perpendicular to ſo many other right Lines AE, BF, CF, given in Poſition, ſhall obtain a given Ratio to one another.* [Vidè Figure 44.]

OF the right Lines given in Poſition, let us ſuppoſe *BF* be produc'd, as alſo its Perpendicular *BD*, till they meet the reſt *AE* and *CF*, viz. *BF* in *E* and *F*, and *BD* in *H* and *G*. Now let $EB = x$, and $EF = a$; and *BF* will be $= a - x$. But ſince, by reaſon of the given Poſition of the right Lines *EF*, *EA*, and *FC*, the Angles *E* and *F*, and conſequently the Proportions of the Sides of the Triangles *EBH* and *FBG* are given. Let *EB* be to *BH* as *d* to *e*; and *BH* will be $= \frac{ex}{d}$, and *EH* ($= \sqrt{EBq + BHq} = \sqrt{xx + \frac{eexx}{dd}}$, that is, $\frac{x}{d} \times \sqrt{dd + ee}$. Let alſo *BF* be to *BG* as *d* to *f*; and *BG* will be $= \frac{fa - fx}{d}$, and *FG* ($= \sqrt{BFq + BGq}$) $=$

$$\sqrt{\frac{aadd - 2axdd + xxdd + ffaa - 2ffax + ffxx}{dd}},$$

that is, $= \frac{a - x}{d}\sqrt{dd + ff}$. Beſides, make $BD = y$, and *HD* will be $= \frac{ex}{d} - y$, and $GD = \frac{fa - fx}{d} - y$; and ſo, ſince $AD$ is $: HD\ (:: EB : EH) :: d : \sqrt{dd + ee}$, and $DC : GD\ (:: BF : FG) :: d : \sqrt{dd + ff}$, *AD* will be $= \frac{ex - dy}{\sqrt{dd + ee}}$, and $DC = \frac{fa - fx - dy}{\sqrt{dd + ff}}$. Laſtly, by reaſon of the given Proportions of the Lines *BD*, *AD*, *DC*, let $BD : AD :: \sqrt{dd + ee} : h - d$, and $\frac{hy - dy}{\sqrt{dd + ee}}$ will be

$(=AD) = \frac{ex - dy}{\sqrt{dd+ee}}$, or $hy = ex$. Let alſo $BD$ : $:: \sqrt{dd+ff} : k - d$, and $\frac{ky - dy}{\sqrt{dd+ff}}$ will be $(= DC$ $\frac{fa - fx - dy}{\sqrt{dd+ff}}$, or $ky = fa - fx$. Therefore $\frac{ex}{h}$ $(=$ $\frac{fa - fx}{k}$; and by Reduction $\frac{fha}{ek+fh} = x$. Wherefor $EB : EF :: h : \frac{ek}{f} + h$, then $BD : EB :: e : h$, and have the Point ſought $D$.

## PROBLEM XXVIII.

*To find the Point D, from which three r Lines DA, D B, D C, drawn to the t. Points, A, B, C, ſhall have a given Rati mong themſelves.* [Vide Figure 45.]

OF the given three Points join any two of them, a poſe $A$ and $C$, and let fall the Perpendicular $BE$ the third $B$, to the Line that conjoins $A$ and $C$, as alſ Perpendicular $DF$ from the Point ſought $D$; and ma $AE = a$, $AC = b$, $EB = c$, $AF = x$, and $FD = y$; $ADq$ will be $= xx + yy$. $FC = b - x$. $CDq$ $(= FC$ $FDq) = bb - 2bx + xx + yy$. $EF = x - a$, and $(= EFq + \overline{EB + FD}^q) = xx - 2ax + aa + cc +$ $+ yy$. Now, ſince $AD$ is to $CD$ in a given Ratio, be as $d$ to $e$; and $CD$ will be $= \frac{e}{d}\sqrt{xx+yy}$. Sinc $AD$ is to $BD$ in a given Ratio, let that be as $d$ to $f$, $BD$ will be $= \frac{f}{d}\sqrt{xx+yy}$. And, conſequently $\frac{eexx + eeyy}{dd}$ will be $(= CDq) = bb - 2bx + xx$ and $\frac{ffxx + ffyy}{dd}$ $(= BDq) = xx - 2ax + aa + cc + 2cy$

which if, for Abbreviation ſake, you write $p$ for $\frac{dd-ee}{d}$, d $q$ for $\frac{dd-ff}{d}$, there will come out $bb-2bx+xx+\frac{p}{d}yy=0$, and $aa+cc-2ax+2cy+\frac{q}{d}xx$ $\frac{q}{d}yy=0$. And by the former you have $\frac{2bqx-bbq}{p}$ $\frac{q}{d}xx+\frac{q}{d}yy$. Wherefore, in the latter, for $\frac{q}{d}xx+yy$, write $\frac{2bqx-bbq}{p}$, and there will come out $\frac{bqx-bbq}{p}+aa+cc-2ax+2cy=0$. Again, for bbreviation ſake, write $m$ for $a-\frac{bq}{p}$, and $2cn$ for $\frac{bbq}{p}$ $-aa-cc$, and you'll have $2mx+2cn=2cy$, and the Terms being divided by $2c$, there ariſes $\frac{mx}{c}+n=y$. Therefore, in the Æquation $bb-2bx+\frac{p}{d}xx+\frac{p}{d}yy$ $=0$, for $yy$ write the Square of $\frac{mx}{c}+n$, and you'll have $b-2bx+\frac{p}{d}xx+\frac{pmm}{dcc}xx+\frac{2pmn}{dc}x+\frac{pnn}{d}=0$. Where, laſtly, if, for Abbreviation ſake, you write $\frac{b}{r}$ for $\frac{p}{d}$ $+\frac{pmm}{dcc}$, and $\frac{sb}{r}$ for $b-\frac{pmn}{dc}$, you'll have $xx=2sx-$ $b-\frac{pnnr}{bd}$, and having extracted the Root $x=s\pm$ $\sqrt{ss-rb-\frac{pnnr}{bd}}$, and having found $x$, the Æquation $\frac{mx}{c}$ $+n=y$ will give $y$; and from $x$ and $y$ given, that is, $AF$ and $FD$, the given Point $D$ is determin'd.

## PROBLEM XXIX.

*To find the Triangle ABC, whoſe three Sides AB, AC, BC, and its Perpendicular DC are in Arithmetical Progreſſion.* [Vide Figure 46.]

MAKE $AC=a$, $BC=x$, and $DC$, the leaſt Line, will be $=2x-a$, and $AB$, the greateſt, will be $=2a-x$. Alſo $AD$ will $(=\sqrt{ACq-DCq})=\sqrt{4ax-4xx}$, and $BD$ $(=\sqrt{BCq-DCq}=\sqrt{4ax-3xx-aa}$. And ſo again, $AB=\sqrt{4ax-4xx}+\sqrt{4ax-3xx-aa}$. Wherefore $2a-x=\sqrt{4ax-4xx}+\sqrt{4ax-3xx-aa}$, or $2a-x-\sqrt{4ax-4xx}=\sqrt{4ax-3xx-aa}$. And the Parts being ſquar'd, $4aa-3xx-4a+2x\times\sqrt{4ax-4xx}=4ax-3xx-aa$, or $5aa-4ax=4a-2x\times\sqrt{4ax-4xx}$. And the Parts being again ſquar'd, and the Terms rightly diſpos'd, $16x^4-80ax^3+144aaxx-104a^3x+25a^4=0$. Divide this Æquation by $2x-a$, and there will ariſe $8x^3-36axx+54aax-25a^3=0$, an Æquation by the Solution whereof $x$ is given from $a$, being any how aſſum'd. $a$ and $x$ being had, make a Triangle, whoſe Sides ſhall be $2a-x$, $a$ and $x$, and a Perpendicular let fall upon the Side $2a-x$, will be $2x-a$.

If I had made the Difference of the Sides of the Triangle to be $d$, and the Perpendicular to be $x$, the Work would have been ſomething neater; this Æquation at laſt coming out, *viz.* $x^3=24ddx-48d^3$.

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## PROBLEM XXX.

*To find a Triangle ABC, whoſe three Sides AB, AC, BC, and the Perpendicular CD ſhall be in a Geometrical Progreſſion.*

MAKE $AC = x$, $BC = a$, and $AB$ will be $= \frac{xx}{a}$. And $CD = \frac{aa}{x}$. And $AD$ $(= \sqrt{ACq - CDq}) = \sqrt{xx - \frac{a^4}{xx}}$; and $BD$ $(= \sqrt{BCq - DCq}) = \sqrt{aa - \frac{a^4}{xx}}$, and conſequently $\frac{xx}{a}$ $(= AB) = \sqrt{xx - \frac{a^4}{xx}} + \sqrt{aa - \frac{a^4}{xx}}$, or $\frac{xx}{a} - \sqrt{aa - \frac{a^4}{xx}} = \sqrt{xx - \frac{a^4}{xx}}$; and the Parts of the Æquation being ſquar'd, $\frac{x^4}{aa} - \frac{2xx}{a} \times \sqrt{aa - \frac{a^4}{xx}} + aa - \frac{a^4}{xx} = xx - \frac{a^4}{xx}$, that is, $x^4 - aaxx + a^4 = 2aax\sqrt{xx - aa}$. And the Parts being again ſquar'd, $x^8 - 2aax^6 + 3a^4x^4 - 2a^6xx + a^8 = 4a^4x^4 - 4a^6xx$. That is, $x^8 - 2aax^6 - a^4x^4 + 2a^6xx + a^8 = 0$. Divide this Æquation by $x^4 - aaxx - a^4$, and there will ariſe $x^4 - aaxx - a^4$. Wherefore $x^4$ is $= aaxx + a^4$. And extracting the Root $xx - \frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}$, or $x = a\sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$. Take therefore $a$, or $BC$, of any Length, and make $BC : AC :: AC : AB :: 1 : \sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$, and the Perpendicular $DC$ of a Triangle $ABC$ made of theſe Sides, will be to the Side $BC$ in the ſame Ratio.

*The ſame otherwiſe.* [Vide Figure 47.]

Since $AB : AC :: BC : DC$. I ſay the Angle $ACB$ is a right one For if you deny it, draw $CE$, making the Angle $ECB$ a right one.

Therefore the Triangles $BCE$, $DBC$ are ſimilar by 8, 6 *Elem.* and conſequently $EB : EC :: BC : DC$, that is, $EB : EC :: AB : AC$. Draw $AF$ perpendicular to $CE$, and by reaſon of the parallel Lines $AF$, $BC$. $EB$ will be $: EC :: AE : FE :: (EB + AE)\ AB : (EC + FE)\ FC$. Therefore by 9, 5 *Elem.* $AC$ is $= FC$, that is, the Hypothenuſe of a right-angled Triangle is equal to the Side, contrary to the 19, 1 *Elem.* Therefore the Angle $ECB$ is not a right one; wherefore it is neceſſary $ACB$ ſhould be a right one. Therefore $ACq + BCq = ABq$. But $ACq = AB \times BC$, therefore $AB \times BC + BCq = ABq$, and extracting the Root $AB = \frac{1}{2}BC + \sqrt{\frac{5}{4}BCq}$. Wherefore take $BC : AB :: 1 : \frac{1 + \sqrt{5}}{2}$, and $AC$ a mean Proportional between $BC$ and $AB$, and $AB : AC :: BC : DC$ will be continually proportional to a Triangle made of theſe Sides.

## PROBLEM XXXI.

*To make the Triangle ABC upon the given Baſe AB, whoſe Vertex C ſhall be in the right Line EC given in Poſition, and the Baſe an Arithmetical Mean between the Sides.* [Vide Figure 48.]

LET the Baſe $AB$ be biſected in $F$, and produc'd till it meet the right Line $EC$ in $E$, and let fall to it the Perpendicular $CD$; and making $AB = a$, $FE = b$, and $BC - AB = x$, $BC$ will be $= a + x$, $AC = a - x$; and by the 13, 2 *Elem.* $BD \left(= \frac{BCq - ACq + ABq}{2AB}\right) = 2x + \frac{1}{2}a$. And conſequently, $FD = 2x$, $DE = b + 2x$, and $CD \left(= \sqrt{CBq - BDq}\right) = \sqrt{\frac{3}{4}aa - 3xx}$. But by reaſon of the given Poſitions of the right Lines $CE$ and $AB$, the Angle $CED$ is given; and conſequently the Ratio of $DE$ to $CD$, which, if it be put as $d$ to $e$, will give the Proportion $d : e :: b + 2x : \sqrt{\frac{3}{4}aa - 3xx}$. Whence the Means and Extremes being multiply'd by each other, there ariſes the Æquation $eb + 2ex = d\sqrt{\frac{3}{4}aa - 3xx}$, the Parts whereof being ſquar'd and rightly order'd, you have $xx =$

$\frac{3}{4}d^2a^2$

$\frac{\frac{9}{4}d^2a^2 - eebb - 4eebx}{4ee+3dd}$, and the Root being extracted $x = \frac{-2eeb + d\sqrt{3eeaa - 3eebb + \frac{9}{4}ddaa}}{4ee+3dd}$. But $x$ being given, there is given $BC = a + x$, and $AC = a - x$.

## PROBLEM XXXII.

*Having the three right Lines AD, AE, BF, given by Poſition, to draw a fourth DF, whoſe Parts DE and EF, intercepted by the former, ſhall be of given Lengths.* [Vide Figure 49.]

LET fall $EG$ perpendicular to $BF$, and draw $EC$ parallel to $AD$, and the three right Lines given by Poſition meeting in $A$, $B$, and $H$, make $AB = a$, $BH = b$, $AH = c$. $ED = D$, $EF = e$, and $HE = x$. Now, by reaſon of the ſimilar Triangles $ABH$, $ECH$, $AH : AB :: HE : EC = \frac{ax}{c}$, and $AH : HB :: HE : CH = \frac{bx}{c}$. Add $HB$, and there comes $CB = \frac{bx+bc}{c}$. Moreover, by reaſon of the ſimilar Triangles $FEC$, $FDB$, $ED$ is : $CB :: EF : CF = \frac{ebx+ebc}{dc}$. Laſtly, by the 12 and 13, 2 *Elem.* you have $\frac{ECq - EFq}{2FC} + \frac{1}{2}FC \; (= CG) = \frac{HEq - ECq}{2CH} - \frac{1}{2}CH$; that is,

$$\frac{\frac{aaxx}{cc} - ee}{\frac{2ebx+2ebc}{dc}} + \frac{ebx+ebc}{2dc} = \frac{xx - \frac{aaxx}{cc}}{\frac{2bx}{c}} - \frac{bx}{2c}.$$ Or

$$\frac{aadxx - eedcc}{ebx+ebc} + \frac{ebx}{d} + \frac{ebc}{d} = \frac{ccx - aax - bbx}{b}.$$

Here, for Abbreviation ſake, for $\frac{cc-aa-bb}{b} - \frac{eb}{d}$ write

$m$,

$m$, and you'll have $\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebc}{d} = mx$; and all the Terms being multiply'd by $x + c$, there will come out $\frac{aadxx - eedc}{eb} - \frac{ebcx}{d} + \frac{ebcc}{d} = mxx + mcx$. Again, for $\frac{aad}{eb} - m$ write $p$; and for $mc + \frac{ebc}{d}$ write $2pq$, and for $-\frac{ebcc}{d} + \frac{eedcc}{eb}$ write $prr$, and $xx$ will become $= 2qx + rr$, and $x = q \pm \sqrt{qq + rr}$. Having found $x$ or $HE$, draw $EC$ parallel to $AB$, and take $FC : BC :: e : d$, and having drawn $FED$, it will ſatisfy the Conditions of the Queſtion.

## PROBLEM XXXIII.

*To a Circle deſcribed from the Center C, and with the Radius CD, to draw a Tangent DB, the Part whereof PB placed between the right Lines given by Poſition, AP and AB ſhall be of a given Length.* [Vide Figure 50.]

FROM the Center $C$ to either of the right Lines given by Poſition, as ſuppoſe to $AB$, let fall the Perpendicular $CE$, and produce it till it meets the Tangent $DB$ in $H$. To the ſame $AB$ let fall alſo the Perpendicular $PG$, and making $EA = a$; $EC = b$, $CD = c$, $BP = d$, and $PG = x$, by reaſon of the ſimilar Triangles $PGB$, $CDH$, you'll have $GB\ (\sqrt{dd - xx}) : PB :: CD : CH = \frac{cd}{\sqrt{dd - xx}}$. Add $EC$, and you'll have $EH = b + \frac{cd}{\sqrt{dd - xx}}$. Moreover, $PG$ is $: GB :: EH : EB = \frac{b}{x}\sqrt{dd - xx} + \frac{cd}{x}$. Moreover, becauſe of the given Angle $PAG$, there is given the Ratio of $PG$ to $AG$, which being made as $e$ to $f$, $AG$ will $= \frac{fx}{e}$. Add $EA$ and $BG$, and you'll have, laſtly, $EB = a$

+

$+\frac{fx}{e}+\sqrt{dd-xx}$. Therefore $\frac{cd}{x}+\frac{b}{x}\sqrt{dd-xx}=a+\frac{fx}{e}+\sqrt{dd-xx}$, and by Tranſpoſition of the Terms, $a+\frac{fx}{e}-\frac{cd}{x}=\frac{b-x}{x}\sqrt{dd-xx}$. And the Parts of the Æquation being ſquar'd, $aa+\frac{2afx}{e}-\frac{2acd}{x}+\frac{ffxx}{ee}-\frac{2cdf}{e}+\frac{ccdd}{xx}=\frac{bbdd}{xx}-bb-\frac{2bdd}{x}+2bx+dd-xx$. And by a due Reduction

$$x^4+\frac{\begin{array}{l}+2aef\\-2bee\end{array}x^3\begin{array}{l}+aaee\\+bbee\\-ddee\\-2cdef\end{array}xx\begin{array}{l}+2bddee\\-2acdee\end{array}x\begin{array}{l}+ccddee\\-bbddee\end{array}}{ee+ff}=0.$$

## PROBLEM XXXIV.

*If a lucid Point,* [*as*] *A, dart forth Rays towards* [*or upon*] *a refracting plain Surface,* [*as*] *C, D; to find the Ray AC, whoſe refracted* [*Part*] *CB ſtrikes the given Point B.* [Vide Figure 51.]

FROM that lucid Point let fall the Perpendicular $AD$ to the refracting Plane, and let the refracted Ray $BC$ meet with it, being produc'd out on both Sides, in $E$; and a Perpendicular let fall from the Point $B$ in $F$, and draw $BD$; and making $AD=a$, $DB=b$, $BF=c$, $DC=x$, make the Ratio of the Sines of Incidence and Refraction, that is, of the Sines of the Angles $CAD$, $CED$, to be $d$ to $e$, and ſince $EC$ and $AC$ (as is known) are in the ſame Ratio, and $AC$ is $\sqrt{aa+xx}$, $EC$ will be $=\frac{d}{e}\sqrt{aa+xx}$. Beſides, $ED\ (=\sqrt{ECq-CDq})=\sqrt{\frac{ddaa+ddxx}{ee}-xx}$, and $DF=\sqrt{bb-cc}$, and $EF=\sqrt{bb-cc}+$

$\sqrt{\frac{ddaa+ddxx}{ee}-xx}$. Laſtly, becauſe of the ſimilar Triangles $ECD$, $EBF$, $ED:DC::EF:FB$, and multiplying the Values of the Means and Extremes into one another, $c\sqrt{\frac{ddaa+ddxx}{ee}-xx}=x\sqrt{bb-cc}+x\times\sqrt{\frac{ddaa+ddxx}{ee}-xx}$, or $\overline{c-x}\sqrt{\frac{ddaa+ddxx}{ee}-xx}=x\sqrt{bb-cc}$, and the Parts of the Æquation being ſquar'd and duly diſpos'd [into Order]

$$x^4-2cx^3=\frac{\begin{array}{l}+ddcc\\+ddaaxx-2ddaacx+ddaacc\\-eebb\end{array}}{dd-ee}=0.$$

## PROBLEM XXXV.

*To find the Locus or Place of the Vertex of a Triangle D, whoſe Baſe AB is given, and the Angles at the Baſe DAB, DBA, have a given Difference.* [Vide Figure 52.]

WHERE the Angle at the Vertex, or (which is the ſame Thing) where the Sum of the Angles at the Baſe is given, 29. 3. *Euclid.* has taught [us], that the Locus [or Place] of the Vertex is in the Circumference of a Circle; but we have propos'd the finding the Place when the Difference of the Angles at the Baſe is given. Let the Angle $DBA$ be greater than the Angle $DAB$, and let $ABF$ be their given Difference, the right Line $BF$ meeting $AD$ in $F$. Moreover, let fall the Perpendicular $DE$ to $BF$, as alſo $DC$ perpendicular to $AB$, and meeting $BF$ in $G$. And making $AB=a$, $AC=x$, and $CD=y$, $BC$ will be $=a-x$. Now ſince in the Triangle $BCG$ there are given all the Angles, there will be given the Ratio of the Sides $BC$ and $GC$, let that be as $d$ to $a$, and $CG$ will $=\frac{aa-ax}{d}$; take away this from $DC$, or $y$, and there will remain $DG=\frac{dy-aa+ax}{d}$. Beſides, becauſe of the ſimilar Triangles $BGC$, and $DGE$, $BG : BC :: DG : DE$. And in the Triangle $BGC$, $a : d :: CG : BC$. And conſequently $aa : dd :: CGq : BCq$, and by compounding $aa+dd : dd :: BGq : BCq$, and extracting the Roots $\sqrt{aa+dd} : d$ (:: $BG : BC$) :: $DG : DE$. Therefore $DE = \frac{dy-aa+ax}{\sqrt{aa+dd}}$.

Moreover, ſince the Angle $ABF$ is the Difference of the Angles $BAD$ and $ABD$, and conſequently the Angles $BAD$ and $FBD$ are equal, the right-angled Triangles $CAD$ and $EBD$ will be ſimilar, and conſequently the Sides proportional [or] $DA : DC :: DB : DE$. But $DC = y$. $DA$ $(= \sqrt{ACq+DCq}) = \sqrt{xx+yy}$. $DB$ $(= \sqrt{BCq+DCq}) = \sqrt{aa-2ax+xx+yy}$, and above $DE$ was $= \frac{dy-aa+ax}{\sqrt{aa+dd}}$. Wherefore $\sqrt{xx+yy} : y :: \sqrt{aa-2ax+xx+yy} : \frac{dy-aa+ax}{\sqrt{aa+dd}}$, and the Squares of the Means and Extremes being multiply'd by each other $aayy-2axyy+xxyy+y^4 = \frac{ddxxyy+ddy^4-2aadxxy}{aa+dd}$ $\frac{-2aady^3+2adyx^3+2adxy^3+a^4x^2+a^4yy-2a^3x^3}{aa+dd}$ $\frac{-2a^3xyy+aax^4+a^2x^2y^2}{aa+dd}$. Multiply all the Terms by $aa+dd$, and reduce thoſe Terms that come out into due Order, and there will ariſe

$$x^4 \begin{matrix} -2a \\ +\frac{2d}{a}y \end{matrix} x^3 \begin{matrix} -2dy \\ +aa \end{matrix} xx \begin{matrix} +\frac{2d}{a}y^3 \\ +\frac{2dd}{a}yy \end{matrix} x \begin{matrix} -ddyy \\ -2dy^3 \\ -y^4 \end{matrix} = 0.$$

Divide this Æquation by $xx-ax\begin{matrix} +dy \\ +yy \end{matrix}$, and there will ariſe $xx \begin{matrix} -a \\ +\frac{2d}{a} \end{matrix} x \begin{matrix} -yy \\ -dy \end{matrix} = 0$; there come out therefore two Æquations in the Solution of this Problem: The firſt, $xx-ax\begin{matrix} +dy \\ +yy \end{matrix} = 0$. is a Circle, *viz.* the Place of the Point $D$, where the Angle $FBD$ is taken on the other Side of the right Line *BF* than what is deſcrib'd in the Figure, the Angle *ABF* being the Sum of the Angles *DAB* and *DBA* at the Baſe, and ſo the Angle *ADB* at the Vertex being given. The laſt, *viz.* $xx\begin{array}{l}-a\\+\frac{2d}{a}y\end{array}x\begin{array}{l}-yy\\-dy\end{array}=0$, is an Hyperbola, the Place of the Point *D*, where the Angle *FBD* obtains the [ſame] Situation from the right Line *BF*, which we deſcrib'd in the Figure; that is, ſo that the Angle *ABF* may be the Difference of the Angles *DAB*, *DBA*, at the Baſe. But this is the Determination of the Hyperbola: Biſect *AB* in *P*; draw *PQ*, making the Angle *BPQ* equal to half the Angle *ABF*: To this draw the Perpendicular *PR*, and *PQ* and *PR* will be the Aſymprotes of this Hyperbola, and *B* a Point through which the Hyperbola will paſs..

Hence ariſes this *Theorem*. Any Diameter, as *AB*, of a right-angled Hyperbola, being drawn, and having drawn the right Lines *AD*, *BD*, *AH*, *BH* from it's Ends to any two Points *D* and *H* of the Hyperbola; theſe right Lines will make equal Angles *DAH*, *DBH* at the Ends of the Diameter.

### *The ſame after a ſhorter Way.* [Vide Figure 53.].

I laid down a Rule about the moſt commodious Election of Terms to proceed with in the Calculus [of Problems] where there happens any Ambiguity in the Election [of ſuch Terms]. Here the Difference of the Angles at the Baſe is indifferent in reſpect to both [or either of the] Angles; and in the Conſtruction of the Scheme, it might equally have been added to the leſſer Angle *DAB*, by drawing from *A* a right Line parallel to *BF*, or ſubtracted from the greater Angle *DBA*, by drawing the right Line *BF*. Wherefore I neither add nor ſubtract it, but add half of it to one of the Angles, and ſubtract half of it from the other. Then ſince it is alſo doubtful whether *AC* or *BC* muſt be made Uſe of for the indefinite Term whereon the Ordinate *DC* ſtands, I uſe neither of them; but I biſect *AB* in *P*, and I make uſe of *PC*; or rather, having drawn *MPQ* making on both Sides the Angles *APQ*, *BPM* equal to half the Difference of the Angles at the Baſe, ſo that it, with the right Lines *AD*, *BD*, may make the Angles *DQP*, *DMP* equal; I let fall to $MQ$ the Perpendiculars $AR$, $BN$, $DO$, and I uſe $DO$ for the Ordinate, and $PO$ for the indefinite Line it ſtands on. I make therefore $PO = x$, $DO = y$, $AR$ or $BN = b$, and $PR$ or $PN = c$. And by reaſon of the ſimilar Triangles $BNM$, $DOM$, $BN$ will be $: DO :: MN : MO$. And by Diviſion [as in the 5th of *Euclid*] $DO - BN, (y-b) : DO\ (y) :: MO - MN\ (ON$ or $c-x) : MO$. Wherefore $MO = \frac{cy-xy}{y-b}$. In like Manner on the other Side, by reaſon of the ſimilar Triangles $ARQ$, $DOQ$, $AR$ will be $: DO :: RQ : QO$, and by Compoſition $DO + AR\ (y+b) : DO\ (y) :: QO + RQ\ (OR$ or $c+x) : QO$. Wherefore $QO = \frac{cy+xy}{y+b}$. Laſtly, by reaſon of the equal Angles $DMQ$, $DQM$, $MO$ and $QO$ are equal, that is, $\frac{cy-xy}{y-b} = \frac{cy+xy}{y+b}$. Divide all by $y$, and multiply by the Denominators, and there will ariſe $cy + cb - xy - xb = cy - cb + xy - xb$, or $cb = xy$, an Æquation that expreſſes (as is commonly known) the Hyperbola.

Moreover, the Locus, or Place of the Point $D$ might have been found without an Algebraick Calculus; for from what we have ſaid above, $DO - BN : ON :: DO : MO\ (QO) :: DO + AR : OR$. That is, $DO - BN : DO + BN :: ON : OR$. And mixtly, $DO : BN :: \frac{ON+OR}{2}\ (NP) : \frac{OR-ON}{2}\ (OP)$. And conſequently, $DO \times OP = BN \times NP$.

## PROBLEM XXXVI.

*To find the Locus or Place of the Vertex of a Triangle whoſe Baſe is given, and one of the Angles at the Baſe differs by a given Angle from [being] double of the other.*

IN the laſt Scheme of the former Problem, let $ABD$ be that Triangle, $AB$ its Baſe biſected in $P$, $APQ$ or $BPM$ half of the given Angle, by which $DBA$ exceeds the double of the Angle $DAB$; and the Angle $DMQ$ will be double of the Angle $DQM$. To $PM$ let fall the Perpendiculars $AR$, $BN$, $DO$, and biſect the Angle $DMQ$ by the right Line $MS$ meeting $DO$ in $S$; and the Triangles $DOQ$, $SOM$ will be ſimilar; and conſequently $OQ : OM :: OD : OS$, and dividing $OQ - OM : OM :: SD : OS ::$ (by the 3. of the 6th *Elem.*) $DM : OM$. Wherefore the 9. of the 5th *Elem.*) $OQ - OM = DM$. Now making $PO = x$, $OD = y$, $AR$ or $BN = b$, and $PR$ or $PN = c$, you'll have, as in the former Problem, $OM = \frac{cy - xy}{y - b}$, and $OQ = \frac{cy + xy}{y + b}$, and conſequently $OQ - OM = \frac{2bcy - 2xyy}{yy - bb}$. Make now $DOq + OMq = DMq$, that is, $yy + \frac{cc - 2cx + xx}{yy - 2by + bb} yy = \frac{4bbcc - 8bcxy + 4xxyy}{y^4 - 2bbyy + b^4} yy$, or $yy + \frac{\overline{cc - 2cx + xx}}{\overline{y - b} \times \overline{y - b}} = \frac{\overline{4bcc - 8bcxy + 4xxyy}}{\overline{y - b} \times \overline{y - b} \times \overline{y + b} \times \overline{y + b}}$; and by due Reduction there will at length ariſe

$$y^4 * \begin{matrix} +cc \\ -2bb \\ -2cx \\ -3xx \end{matrix} yy \begin{matrix} +2bxx \\ +4bcx \\ +2bcc \end{matrix} y \begin{matrix} +b^4 \\ -3bbcc \\ -2bbcx \\ +bbxx \end{matrix} = 0.$$

Which gives the Relation of the Curve: Which becomes an Hyperbola when the Angle $BPM$ (vaniſhes, or) becomes nothing; or, which is the ſame Thing, when one of the Angles at the Baſe $DBA$ is double of the other $DAB$. For then $BN$ or $b$ vaniſhing, the Æquation will become $yy = 3xx + 2cx - cc$.

And from the Conſtruction of this Æquation there comes this *Theorem.* [*Vide Figure* 54.] If from the Center $C$, the Aſymptotes being $CS$, $CT$, containing the Angle $SCT$ of 120 Degrees, you deſcribe any Hyperbola, as $DV$, whoſe Semi-Axis are $CV$, $CA$; produce $CV$ to $B$, ſo that $VB$ ſhall $= VC$, and from $A$ and $B$ you draw any how the right Lines $AD$, $BD$, meeting at the Hyperbola; the Angle $BAD$ will be half the Angle $ABD$, but a third Part of the Angle $ADE$, which the right Line $AD$ comprehends together with $BD$ produc'd. This is to be underſtood of an Hyperbola that paſſes thro' the Point $V$. Now if the two right Lines $Ad$ and $Bd$, drawn from the ſame Points $A$ and $B$, meet in the conjugate Hyperbola that paſſes through $A$, then of thoſe two external Angles of the Triangle at the Baſe, that at *B* will be double of that at *A*.

## PROBLEM XXXVII.

*To deſcribe a Circle through two given Points that ſhall touch a right Line given by Poſition.* [Vide Figure 55.]

LET *A* and *B* be the two Points, and *EF* the right Line given by Poſition, and let it be requir'd to deſcribe a Circle *ABE* through thoſe Points which ſhall touch that right Line *FE*. Join *AB* and biſect it in *D*. Upon *D* erect the Perpendicular *DF* meeting the right Line *FE* in *F*, and the Center of the Circle will fall upon this laſt drawn Line *DF*, as ſuppoſe in *C*. Join therefore *CB*; and on *FE* let fall the Perpendicular *CE*, and *E* will be the Point of Contact, and *CB* and *CE* equal, as being Radii of the Circle ſought. Now ſince the Points *A*, *B*, *D*, and *F*, are given, let $DB = a$, and $DF = b$; and ſeek for *DC* to determine the Center of the Circle, which therefore call $x$. Now in the Triangle *CDB*, becauſe the Angle at *D* is a right one, you have $\sqrt{DBq + DCq}$, that is, $\sqrt{aa + xx} = CB$. Alſo $DF - DC$, or $b - x = CF$. And ſince in the right-angled Triangle *CFE* the Angles are given, there will be given the Ratio of the Sides *CF* and *CE*. Let that be as $d$ to $e$; and *CE* will be $= \frac{e}{d} \times CF$, that is, $= \frac{eb - ex}{d}$. Now put [or make] *CB* and *CE* (the Radii of the Circle ſought) equal to one another, and you'll have the Æquation $\sqrt{aa + xx} = \frac{eb - ex}{d}$. Whoſe Parts being ſquar'd and multiply'd by $dd$, there ariſes. $aadd + ddxx = eebb - 2eebx + eexx$; or $xx = \frac{-2eebx - aadd + eebb}{dd - ee}$.

And extracting the Root $x = \frac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$.

Therefore the Length of *DC*, and conſequently the Center *C* is found, from which a Circle is to be deſcrib'd through the Points *A* and *B* that ſhall touch the right Line *FE*.

PRO-

## PROBLEM XXXVIII.

### *To deſcribe a Circle through a given Point that ſhall touch two right Lines given by Poſition.* [Vide Figure 56.]

N. B. *This Propoſition is reſolv'd as Prop. 37. for the Point A being given, there is alſo given the other Point B.*

SUPPOSE the given Point to be *A*; and let *EF*, *FG*, be the two right Lines given by Poſition, and *AEG* the Circle ſought touching the ſame, and paſſing through that Point *A*. Let the Angle *EFG* be biſected by the right Line *CF*, and the Center of the Circle will be found therein. Let that be *C*; and having let fall the Perpendiculars *CE*, *CG* to *EF* and *FG*, *E* and *G* will be the Points of Contact. Now in the Triangles *CEF*, *CGF*, ſince the Angles *E* and *G* are right ones, and the Angles at *F* are halves of the Angle *EFG*, all the Angles are given, and conſequently the Ratio of the Side *CF* to *CE* or *CG*. Let that be as $d$ to $e$; and if for determining the Center of the Circle ſought *C*, there be aſſum'd $CF = x$, *CE* or *CG* will be $= \frac{ex}{d}$. Beſides, let fall the Perpendicular *AH* to *FC*, and ſince the Point *A* is given, the right Lines *AH* and *FH* will be given. Let them be call'd $a$ and $b$, and taking *FC* or $x$ from *FH* or $b$, there will remain $CH = b - x$: To whoſe Square $bb - 2bx + xx$ add the Square of *AH* or $aa$, and the Sum $aa + bb - 2bx + xx$ will be $ACq$ by the 47. 1. *Eucl.* becauſe the Angle *AHC* is, by Suppoſition, a right one. Now make the Radii of the Circle *AC* and *CG* equal to each other; that is, make an Equality between their Values, or between their Squares, and you'll have the Æquation $aa + bb - 2bx + xx = \frac{eexx}{dd}$. Take away $xx$ from both Sides, and changing all the Signs, you'll have $-aa - bb + 2bx = xx - \frac{eexx}{dd}$. Multiply all by $dd$, and divide by $dd - ee$, and it will become

$\frac{-aadd-bbdd+2bddx}{dd-ee}=xx$. The Root of which Æquation being extracted, is $x=\frac{bdd-d\sqrt{eebb+eeaa-ddaa}}{dd-ee}$. Therefore the Length of $FC$ is found, and conſequently the Point $C$, which is the Center of the Circle ſought.

If the found Value $x$, or $FC$, be taken from $b$, or $HF$, there will remain $HC=\frac{-eeb+d\sqrt{eebb+eeaa-ddaa}}{dd-ee}$ the ſame Æquation which came out in the former Problem, for determining the Length of $DC$.

## PROBLEM XXXIX.

### *To deſcribe a Circle through two given Points, which ſhall touch another Circle given by Poſition.* [Vide Problem 11, and Figure 57.]

LET $AB$, be the two Points given, $EK$ the Circle given by Magnitude and Poſition, $F$ its Center, $ABE$ the Circle ſought, paſſing through the Points $A$ and $B$, and touching the other Circle in $E$, and let $C$ be its Center. Let fall the Perpendiculars $CD$ and $FG$ to $AB$ being produc'd, and draw $CF$ cutting the Circles in the Point of Contact $E$, and draw alſo $FH$ parallel to $DG$, and meeting $CD$ in $H$. Theſe being [thus] conſtructed, make $AD$ or $DB=a$, $DG$ or $HF=b$, $GF=c$, and $EF$ (the Radius of the Circle given) $=d$, and $DC=x$; and $CH$ will be $(=CD-FG)=x-c$, and $CFq\ (=CHq+HFq)=xx-2cx+cc+bb$, and $CBq\ (=CDq+DBq)=xx+aa$, and conſequently $CB$ or $CE=\sqrt{xx+aa}$. To this add $EF$, and you'll have $CF=d+\sqrt{xx+aa}$, whoſe Square $dd+aa+xx+2d\sqrt{xx+aa}$, is $=$ to the Value of the ſame $CFq$ found before, *viz.* $xx-2cx+cc+bb$. Take away from both Sides $xx$, and there will remain $dd+aa+2d\sqrt{xx+aa}=cc+bb-2cx$. Take away moreover $dd+aa$, and there will come out $2d\sqrt{xx+aa}=cc+bb-dd-aa-2cx$. Now, for Abbreviation ſake, for

cc+

$ec + bb - dd - aa$, write $2gg$, and you'll have $2d\sqrt{xx+aa} = 2gg - 2cx$, or $d\sqrt{xx+aa} = gg - cx$. And the Parts of the Æquation being ſquar'd, there will come out $ddxx + ddaa = g^4 - 2ggcx + ccxx$. Take from both Sides $ddaa$ and $ccxx$, and there will remain $ddxx - ccxx = g^4 - ddaa - 2ggcx$. And the Parts of the Æquation being divided by $dd - cc$, you'll have $xx = \frac{g^4 - ddaa - 2ggcx}{dd - cc}$. And by Extraction of the affected Root $x = \frac{-ggc + \sqrt{g^4dd - d^4aa + ddaacc}}{dd - cc}$.

Having found therefore $x$, or the Length of $DC$, biſect $AB$ in $D$, and at $D$ erect the Perpendicular $DC = \frac{-ggc + d\sqrt{g^4 - aadd + aacc}}{dd - cc}$. Then from the Center $C$, through the Point $A$ or $B$, deſcribe the Circle $ABE$; for that will touch the other Circle $EK$, and paſs through both the Points $A$, $B$. Q. E. F.

## PROBLEM XL.

### *To deſcribe a Circle through a given Point which ſhall touch a given Circle, and alſo a right Line, both given in Poſition,* [Vide Figure 58.]

LET the Circle to be deſcrib'd be $BD$, its Center $C$, and $B$ a Point through which it is to be deſcrib'd, and $AD$ the right Line which it ſhall touch; the Point of Contact $D$, and the Circle which it ſhall touch $GEM$, its Center $F$, and its Point of Contact $E$. Produce $CD$ to $Q$, ſo that $DQ$ ſhall be $= EF$, and through $Q$ draw $QN$ parallel to $AD$. Laſtly, from $B$ and $F$ to $AD$ and $QN$, let fall the Perpendiculars $BA$, $FN$; and from $C$ to $AB$ and $FN$ let fall the Perpendiculars $CK$, $CL$. And ſince $BC = CD$, or $AK$, $BK$ will be $(= AB - AK) = AB - BC$, and conſequently $BKq = ABq - AB \times BC + BCq$. Subtract this from $BCq$, and there will remain $2AB \times BC - ABq$ for the Square of $CK$. Therefore $AB \times \overline{2BC - AB} = CKq$; and for the ſame Reaſon $FN \times \overline{2FC - FN} = CLq$.

$CLq$, and conſequently $\frac{CKq}{AB}+AB=2BC$, and $\frac{CLq}{FN}+FN=2FC$. Wherefore, if for $AB$, $CK$, $FN$, $KL$, and $CL$, you write $a$, $y$, $b$, $c$, and $c-y$, you'll have $\frac{yy}{2a}+\frac{1}{2}a=BC$, and $\frac{cc-2cy+yy}{2b}+\frac{1}{2}b=FC$. From $FC$ take away $BC$, and there remains $EF=\frac{cc-2cy+yy}{2b}+\frac{1}{2}b-\frac{yy}{2a}-\frac{1}{2}a$. Now, if the Points where $FN$ being produc'd cuts the right Line $AD$, and the Circle $GEM$ be mark'd with the Letters $H$, $G$, and $M$, and upon $HG$ produc'd you take $HR=AB$, ſince $HN$ $(=DQ=EF)$ is $=GF$, by adding $FH$ on both Sides, you'll have $FN=GH$, and conſequently $AB-FN$ $(=HR-GH)$ $=GR$, and $AB-FN+2EF$; that is, $a-b+2EF=RM$, and $\frac{1}{2}a-\frac{1}{2}b+EF=\frac{1}{2}RM$. Wherefore, ſince above $EF$ was $=\frac{cc-2cy+yy}{2b}+\frac{1}{2}b-\frac{yy}{aa}-\frac{1}{2}a$, if this be written for $EF$ you'll have $\frac{1}{2}RM=\frac{cc-2cy+yy}{2b}-\frac{yy}{2a}$. Call therefore $RM$ $d$, and $d$ will be $=\frac{cc-2cy+yy}{b}-\frac{yy}{a}$. Multiply all the Terms by $a$ and $b$, and there will ariſe $abd=acc-2acy+ayy-byy$. Take away from both Sides $acc-2acy$, and there will remain $abd-acc+2acy=ayy-byy$. Divide by $a-b$, and there will ariſe $\frac{abd-acc+2acy}{a-b}=yy$. And extracting the Root $y=\frac{ac}{a-b}\pm\sqrt{\frac{aabd-abbd+abcc}{aa-2ab+bb}}$. Which Concluſions may be thus abbreviated; make $c:b::d:e$, then $a-b:a::c:f$; and $fe-fc+2fy$ will be $=yy$, or $y=f\pm\sqrt{ff+fe-fc}$. Having found $y$, or $KC$, or $AD$, take $AD=f\pm\sqrt{ff+fe-fc}$, and at $D$ erect the Perpendicular $DC$ $(=BC)=\frac{KCq}{2AB}+\frac{1}{2}AB$; and from the Center $C$, at the Interval $CB$ or $CD$, deſcribe the Circle $BDE$, for this paſſing through the given Point $B$, will touch the right Line $AD$ in $D$, and the Circle $GEM$ in $E$. Q. E. F.

Hence alſo a Circle may be deſcrib'd which ſhall touch two given Circles, and a right Line given by Poſition. [*Vide Figure* 59.] For let the given Circles be $RT$, $SV$, their Centers $B$ $F$, and the right Line given by Poſition $PQ$. From the Center $F$, with the Radius $FS - BR$, deſcribe the Circle $EM$. From the Point $B$ to the right Line $PQ$ let fall the Perpendicular $BP$, and having produc'd it to $A$, ſo that $PA$ ſhall be $= BR$, through $A$ draw $AH$ parallel to $PQ$, and deſcribe a Circle which ſhall paſs through the Point $B$, and touch the right Line $AH$ and the Circle $EM$. Let its Center be $C$; join $BC$, cutting the Circle $RT$ in $R$, and the Circle $RS$ deſcrib'd from the ſame Center $C$, and the Radius $CR$ will touch the Circles $RT$, $SV$, and the right Line $PQ$, as is manifeſt by the Conſtruction.

## PROBLEM XLI.

### *To deſcribe a Circle that ſhall paſs through a given Point, and touch two other Circles given in Poſition and Magnitude.* [Vide Figure 60.]

LET the given Point be $A$, and let the Circles given in Magnitude and Poſition be $TIV$, $RHS$, their Centers $C$ and $B$; the Circle to be deſcrib'd $AIH$, its Center $D$, and the Points of Contact $I$ and $H$. Join $AB$, $AC$, $AD$, $DB$, and let $AB$ produc'd cut the Circle $RHS$ in the Points $R$ and $S$, and $AC$ produc'd, cut the Circle $TIV$ in $T$ and $V$. And having let fall the Perpendiculars $DE$ from the Point $D$ to $AB$, and $DF$ from the Point $D$ to $AC$ meeting $AB$ in $G$, and [alſo the Perpendicular] $CK$ to $AB$; in the Triangle $ADB$, $ADq - DBq + ABq$ will be $= 2AE \times AB$, by the 13th of the 2*d. Elem.* But $DB = AD + BR$, and conſequently $DBq = ADq + 2AD \times BR + BRq$. Take away this from $ADq + ABq$, and there will remain $ABq - 2AD \times BR - BRq$ for $2AE \times AB$. Moreover, $ABq - BRq$ is $= \overline{AB - BR} \times \overline{AB + BR} = AR \times AS$. Wherefore, $AR \times AS - 2AD \times BR = 2AE \times AB$. And $\frac{AR \times AS - 2AB \times AE}{BR} = 2AD$

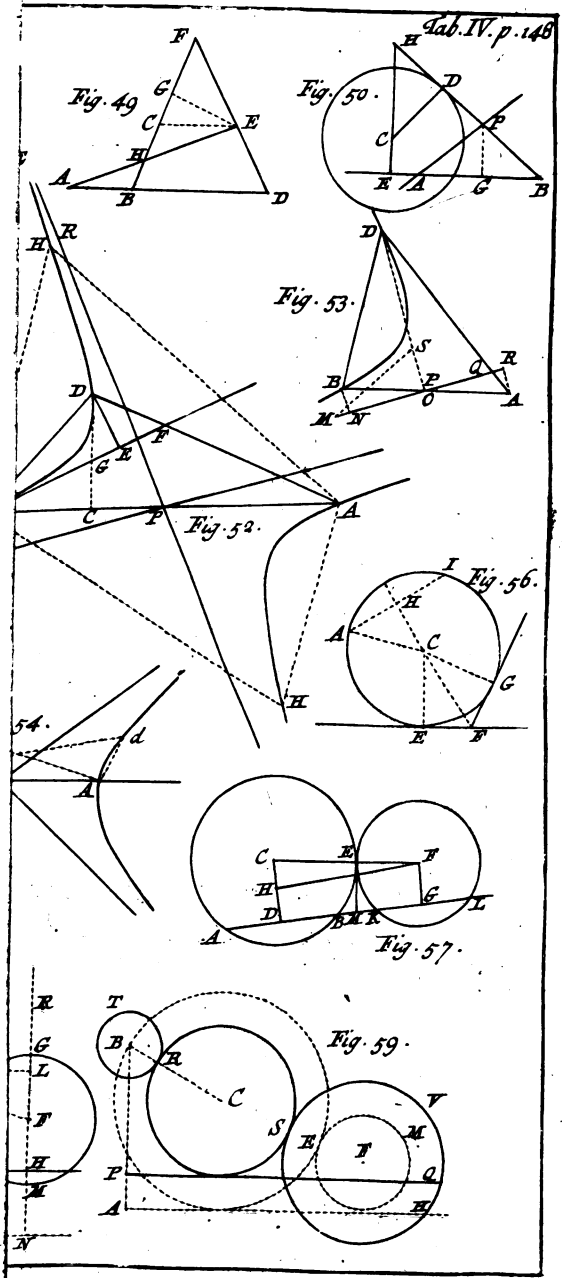
Fig. 49.
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Fig. 53.
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A
H

$=2AD$. And by a like Reaſoning in the Triangle $ADC$, there will come out again $2AD = \frac{TAV - 2CAF}{CT}$. Wherefore $\frac{RAS - 2BAE}{BR} = \frac{TAV - 2CAF}{CT}$. And $\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR} = \frac{2CAF}{CT}$. And $\overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AC} = AF$. Whence ſince $AK : AC :: AF : AG$, $AG$ will $= \overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AK}$. Take away this from $AE$, or $\frac{2KAE}{CT} \times \frac{CT}{2AK}$ and there will remain $GE = \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2AK}$. Whence ſince $KC : AK :: GE : DE$; $DE$ will be $= \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2KC}$. Upon $AB$ take $AP$, which let be to $AB$ as $CT$ to $BR$, and $\frac{2PAE}{CT}$ will be $= \frac{2BAE}{BR}$, and ſo $\frac{2PK \times AE}{CT} = \frac{2BAE}{BR} - \frac{2KAE}{CT}$, and ſo $DE = \frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2PK \times AE}{CT} \times \frac{CT}{2KC}$. Upon $AB$ erect the Perpendicular $AQ = \frac{RAS}{BR} - \frac{TAV}{CT} \times \frac{CT}{2KC}$, and in it take $QO = \frac{PK \times AE}{KC}$, and $AO$ will be $=DE$. Join $DO$, $DQ$, and $CP$, and the Triangles $DOQ$, $CKP$, will be ſimilar, becauſe their Angles at $O$ and $K$ are right ones, and the Sides ($KC : PK :: AE$, or $DO : QO$) proportional. Therefore the Angles $OQD$, $KPC$, are equal, and conſequently $QD$ is perpendicular to $CP$. Wherefore if $AN$ be drawn parallel to $CP$, and meeting $QD$ in $N$, the Angle $ANQ$ will be a right one, and the Triangles $AQN$, $PCK$ ſimilar; and conſequently $PC : KC :: AQ : AN$. Whence ſince $AQ$ is $\frac{RAS}{BR}$

$\frac{RAS}{BR} - \frac{TAV}{CT} \times \frac{CT}{2KC}$, $AN$ will be $\frac{RAS}{BR} - \frac{TAV}{CT} \times \frac{CT}{2PC}$. Produce $AN$ to $M$, ſo that $NM$ ſhall be $= AN$, and $AD$ will $= DM$, and conſequently the Circle will paſs through the Point $M$.

Since therefore the Point $M$ is given, there follows this Reſolution of the Problem, without any farther Analyſis.

Upon $AB$ take $AP$, which muſt be to $AB$ as $CT$ to $BR$; join $CP$, and draw parallel to it $AM$, which ſhall be to $\frac{RAS}{BR} - \frac{TAV}{CT}$, as $CT$ to $PC$; and by the Help of the 39th *Probl.* deſcribe through the Points $A$ and $M$ the Circle $AIHM$, which ſhall touch either of the Circles $TIV$, $RHS$, and the ſame Circle ſhall touch both. Q. E. F.

And hence alſo a Circle may be deſcrib'd, which ſhall touch three Circles given in Magnitude and Poſition. For let the Radii of the given Circles be $A$, $B$, $C$, and their Centers $D$, $E$, $F$. From the Centers $E$ and $F$, with the Radii $B \pm A$ and $C \pm A$ deſcribe two Circles, and let a third Circle which touches theſe [two] be alſo deſcrib'd, and let it paſs through the Point $A$; let its Radius be $G$, and its Center $H$, and a Circle deſcrib'd on the ſame Center $H$, with the Radius $G \pm A$, ſhall touch the three former Circles, as was requir'd.

PRO-

## PROBLEM XLII.

*Three Staves being erected, or ſet up an End, in ſome certain Part of the Earth perpendicular to the Plane of the Horizon, in the Points A, B, and C, whereof that which is in A is ſix Foot long, that in B eighteen, and that in C eight, the Line AB being thirty Foot long; it happens on a certain Day [in the Year] that the End of the Shadow of the Staff A paſſes through the Points B and C, and of the Staff B through A and C, and of the Staff C through the Point A. To find the Sun's Declination, and the Elevation of the Pole, or the Day and Place where this ſhall happen.* [Vide Figure 61.]

BEcauſe the Shadow of each Staff deſcribes a Conick Section, or the Section of a luminous Cone, whoſe Vertex is the Top of the Staff; I will feign *BCDEF* to be ſuch a Curve, [whether it be an Hyperbola, Parabola, or Ellipſe] as the Shadow of the Staff *A* deſcribes that Day, by putting *AD*, *AE*, *AF*, to have been its Shadows, when *BC*, *BA*, *CA*, were reſpectively the Shadows of the Staves *B* and C. And beſides I will ſuppoſe *PAQ* to be the Meridional Line, or the Axis of this Curve, to which the Perpendiculars *BM*, *CH*, *DK*, *EN*, and *PL*, being let fall, are Ordinates. And I will denote theſe Ordinates indefinitely [or indifferently] by the Letter $y$, and the intercepted Parts of the Axis *AM*, *AH*, *AK*, *AN*, and *AL* by the Letter $x$. I'll ſuppoſe, laſtly, the Æquation $aa \perp bx \perp cxx = yy$, to expreſs the Relation of $x$ and $y$, (*i. e.* the Nature of the Curve) aſſuming $aa$, $b$, and $c$, as known Quantities, as they will be found to be from the Analyſis. Where I made the unknown Quantities of two Dimenſions only becauſe the Æquation is [to expreſs] a Conick Section; and I omitted the odd Dimenſions of $y$, becauſe it is an Ordinate to the Axis. And I denoted the Signs of $b$ and $c$, as being indeterminate by the Note $\perp$, which I uſe indifferently for $+$ or $-$, and its oppoſite $\mp$ for the contrary. But I made the Sign of the Square $aa$ Affirmative, becauſe the concave Part of the Curve neceſſarily contains the Staff *A*, projecting its Shadows to the oppoſite Parts (*C* and *F*, *D* and *E*); and then, if at the Point *A* you erect the Perpendicular $A\beta$, this will ſome where meet the Curve, ſuppoſe in $\beta$, that is, the Ordinate $y$, where $x$ is nothing, will [ſtill] be real. From thence it follows that its Square, which in that Caſe is $aa$, will be Affirmative.

It is manifeſt therefore, that this fictitious Æquation $aa \perp bx \perp cxx = yy$, as it is not fill'd with ſuperfluous Terms, ſo neither is it more reſtrain'd [or narrower] than what is capable of ſatisfying all the Conditions of the Problem, and will denote the Hyperbola, Ellipſe, or Parabola, according as the Values of $aa$, $b$, $c$, ſhall be determin'd, or found to be nothing but what may be their Value; and with what Signs $b$ and $c$ are to be affected, and thence what Sort of a Curve this may be, will be manifeſt from the following Analyſis.

## *The former Part of the Analyſis.*

Since the Shadows are as the Altitude of the Staves, you'll have $BC : AD :: AB : AE$ $(:: 18 : 6) :: 3 : 1$. Alſo $CA : AF$ $(:: 8 : 6) :: 4 : 3$. Wherefore naming (or making] $AM = +r$, $MB = \perp s$, $AH = \perp t$, and $HC = +v$. From the Similitude of the Triangles $AMB$, $ANE$, and $AHC$, $ALF$, $AN$ will be $= -\frac{r}{3}$. $NE = \top\frac{s}{3}$. $AL = \top\frac{3t}{4}$, and $LF = -\frac{3v}{4}$; whoſe Signs I put contrary to the Signs of $AM$, $MB$, $AH$, $HC$, becauſe they tend contrary Ways with reſpect to the Point *A* from which they are drawn, and from the Axis $PQ$ on which they ſtand. Now theſe being reſpectively written for $x$ and $y$ in the fictitious Æquation $aa \perp bx \perp cxx = yy$.

$r$ and $\perp s$ will give $aa \perp br \perp crr = ss$.

$-\frac{r}{3}$ and $\top\frac{s}{3}$ will give $aa \top \frac{br}{3} \perp \frac{1}{9}crr = \frac{1}{9}ss$.

$\perp t$ and $+v$ will give $aa \perp b \times \perp t \perp ctt = vv$.

$\top\frac{3}{4}t$ and $-\frac{3}{4}v$ will give $aa \underline{\perp} \frac{3}{4}b \times \top t \perp \frac{9}{16}ctt = \frac{9}{16}vv$.

Now, by exterminating $ss$ from the firſt and ſecond Æquations, in order to obtain $r$, there comes out $\frac{2aa}{\perp b} = r$. Whence it is manifeſt, that $\perp b$ is Affirmative, becauſe $r$ is ſo. Alſo by exterminating $vv$ from the third and fourth, (after having written for $\perp b$ its Value $+b$) to obtain $t$, there comes out $\frac{aa}{3b} = \perp t$, therefore $t$ is poſitive and equal to $\frac{aa}{3b}$, and having writ $\frac{2aa}{b}$ for $r$ in the firſt, and $\frac{aa}{3b}$ for $t$ in the third, there ariſe $3aa \perp \frac{4a^4c}{bb} = ss$, and $\frac{4}{3}aa \perp \frac{a^4c}{9bb} = vv$.

Moreover, having let fall $B\lambda$ perpendicular upon $CH$, $BC$ will be $: AD$ $(: : 3 : 1) : : B\lambda : AK : : C\lambda : DK$. Wherefore, ſince $B\lambda$ is $(= AM - AH = r - t) = \frac{5aa}{3b}$, $AK$ will be $= \frac{5aa}{9b}$, but with a Negative Sign, *viz.* $-\frac{5aa}{9b}$. Alſo ſince $C\lambda$ $(= CH \perp BM = v \perp s) = \sqrt{\frac{4aa}{3} \perp \frac{a^4c}{9bb}} \perp \sqrt{3aa \perp \frac{4a^4c}{bb}}$, and therefore $DK$ $(= \frac{1}{3}C\lambda) = \sqrt{\frac{4aa}{27} \perp \frac{a^4c}{81bb}} \perp \sqrt{\frac{1}{3}aa \perp \frac{4a^4c}{9bb}}$; which being reſpectively written in the Æquation $aa \perp bx \perp cxx = yy$, or rather the Æquation $aa + bx \perp cxx = yy$, becauſe $b$ hath before been found to be Poſitive, for $AK$ and $DK$, or $x$ and $y$, there comes out $\frac{4aa}{9} \perp \frac{25a^4c}{81bb} = \frac{13}{27}aa \perp \frac{37a^4c}{81bb} \perp 2\sqrt{\frac{4aa}{27} \perp \frac{a^4c}{81bb}} \times \sqrt{\frac{aa}{3} \perp \frac{4a^4c}{9bb}}$. And by Reduction $-bb \top 4aac = \perp 2\sqrt{36b^4 \perp 51aabbc + 4a^4cc}$; and the Parts being ſquar'd, and again reduc'd, there comes out $0 = 143b^4 \perp 196aabbc$, or $\frac{-143bb}{196aa} = \perp c$. Whence it is manifeſt, that $\perp c$ is Negative, and conſequently the fictitious Æquation $aa \perp bx \perp cxx = yy$ will be of this Form, $aa + bx - cxx = yy$. And its Center and two Axes are thus found.

Making $y = 0$, as happens in the Vertex's of the Figure $P$ and $Q$, you'll have $aa + bx = cxx$, and having extracted the Root $x = \frac{b}{2c} \pm \sqrt{\frac{bb}{4cc} + \frac{aa}{c}} = \left\{\begin{matrix} AQ \\ AP \end{matrix}\right\}$, that is, $AQ = \frac{b}{2c} + \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$, and $AP = \frac{b}{2c} - \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$, where $AP$ and $AQ$ are computed from $A$ towards the Parts $Q$; and conſequently when $AP$ is computed from $A$ towards $P$, its Value will be found to be $-\frac{b}{2c} + \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$. And conſequently, taking $AV = \frac{b}{2c}$, $V$ will be the Center of the Ellipſe, and $VQ$, or $VP$, $\left(\sqrt{\frac{bb}{4cc} + \frac{aa}{c}}\right)$ the greateſt Semi-Axis. If, moreover, the Value of $AV$, or $\frac{b}{2c}$, be put for $x$ in the Æquation $aa + bx - cxx = yy$, there will come out $aa + \frac{bb}{4c} = yy$. Wherefore $aa + \frac{bb}{4c}$ will be $= VZq$, that is, to the Square of the leaſt Semi-Axis. Laſtly, in the Values of $AV$ and $VQ$, $VZ$ already found, writing $\frac{143bb}{196aa}$ for $c$, there come out $\frac{98aa}{143b} = AV$, $\frac{112aa\sqrt{3}}{143b} = VQ$, and $\frac{8a\sqrt{3}}{\sqrt{143}} = VZ$.

### *The other Part of the Analyſis.* [Vide Figure 62.]

Suppoſe now the Staff $AR$ ſtanding on the Point $A$, and $RPQ$ will be the Meridional Plane, and $RPZQ$ the luminous Cone whoſe Vertex is $R$. Let moreover $TXZ$ be a Plane cutting the Horizon in $VZ$, and the Meridional Plane in $TVX$, which Section let it be perpendicular to the Axis of the World, or of the Cone, and it will cut the Cone in the Periphery of the Circle $TZX$, which will be every where at an equal Diſtance, as $RX$, $RZ$, $RT$, from its Vertex. Wherefore, if $PS$ be drawn parallel to $TX$, you'll have $RS = RP$, by reaſon of the equal Quantities $RX$, $RT$; and alſo $SX = XQ$, by reaſon of the equal Quantities $PV$, $VQ$; whence $RX$ or $RZ$ $\left(= \frac{RS+RQ}{2}\right) = \frac{RP+RQ}{2}$. Laſtly, draw $RV$, and ſince $VZ$ perpendicularly ſtands on the Plane $RPQ$, (as being the Section of the Planes perpendicularly ſtanding on the ſame [Plane]) the Triangle $RVZ$ will be right-angled at $V$.

Now making $RA = d$, $AV = e$, $VP$ or $VQ = f$, and $VZ = g$, you'll have $AP = f - e$, and $RP = \sqrt{ff - 2ef + ee + dd}$. Alſo $AQ = f + e$, and $RQ = \sqrt{ff + 2ef + ee + dd}$; and conſequently $RZ$ $\left(= \frac{RP+RQ}{2}\right) = \frac{\sqrt{ff-2ef+ee+dd} + \sqrt{ff+2ef+ee+dd}}{2}$. Whoſe Square $\frac{dd+ee+ff}{2} + \frac{1}{2}\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4}$, is equal $(RVq + VZq = RAq + AVq + VZq)$ to $dd + ee + gg$. Now having reduc'd $\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4} = dd + ee - ff + 2gg$, and the Parts being ſquar'd and reduc'd into Order, $ddff = ddgg + eegg - ffgg + g^4$, or $\frac{ddff}{gg} = dd + ee - ff + gg$. Laſtly, 6, $\frac{98aa}{143b}$, $\frac{112aa\sqrt{3}}{143b}$, $\frac{8a\sqrt{3}}{\sqrt{143}}$ (the Values of $AR$, $AV$, $VQ$, and $VZ$) being reſtor'd for $d$, $e$, $f$, and $g$, there ariſes $36 - \frac{196a^4}{143bb} + \frac{192aa}{143} = \frac{36, 14, 14aa}{143bb}$, and thence by Reduction $\frac{49a^4 + 36,49aa}{48aa + 1287} = bb$.

In the firſt Scheme $AMq + MBq = ABq$, that is, $rr + ss = 33 \times 33$. But $r$ was $= \frac{2aa}{b}$, and $ss = 3aa -$

$\frac{4a^4c}{bb}$, whence $rr=\frac{4a^4}{bb}$, and ſubſtituting $\frac{143bb}{196aa}$ for $c$) $ss=\frac{4aa}{49}$. Wherefore $\frac{4a^4}{bb}+\frac{4aa}{49}=33\times33$, and thence by Reduction there again reſults $\frac{4{,}49a^4}{53361-4aa}=bb$. Putting therefore an Equality between the two Values of $bb$, and dividing each Part of the Æquation by 49, you'll have $\frac{a^4+36aa}{48aa+1287}=\frac{4a^4}{53361-4aa}$; whoſe Parts being multiply'd croſs-ways, and divided by 49, there comes out $4a^4=981aa+274428$, whoſe Root $aa$ is $\frac{981+\sqrt{1589625}}{8}=280{,}2254144$.

Above was found $\frac{4{,}49a^4}{53361-4aa}=bb$, or $\frac{14aa}{\sqrt{53361-4aa}}=b$. Whence $AV$ $\left(\frac{98aa}{143b}\right)$ is $\frac{7\sqrt{53361-4aa}}{143}$, and $VP$, or $VQ$ $\left(\frac{112aa\sqrt{3}}{143b}\right)$ is $\frac{8}{143}\sqrt{160083-12aa}$. That is, by ſubſtituting 280,2254144 for $aa$, and reducing the Terms into Decimals, $AV=11{,}188297$, and $VP$ or $VQ=22{,}147085$; and conſequently $AP$ $(PV-AV)=10{,}958788$, and $AQ$ $(AV+VQ)$ 33,335382.

Laſtly, if $\frac{1}{6}AR$ or 1 be made Radius, $\frac{1}{6}AQ$ or 5,355897 will be the Tangent of the Angle $ARQ$ of 79 *gr.* 47′. 48″. and $\frac{1}{6}AP$ or 1,826465 the Tangent of the Angle $ARP$ of 61 *gr.* 17′. 52″. half the Sum of which Angles 70 *gr.* 32′. 50″. is the Complement of the Sun's Declination; and the Semi-difference 9 *gr.* 14′. 58″. the Complement of the Latitude of the Place. Then, the Sun's Declination was 19 *gr.* 27′. 10″. and the Latitude of the Place 80 *gr.* 45′. 20″. which were to be found.

# PROBLEM XLIII.

*If at the Ends of the Thread DAE, moving upon the fix'd Tack A, there are hang'd two Weights, D and E, whereof the Weight E ſlides through the oblique Line BG given in Poſition; to find the Place of the Weight E, where theſe Weights are in Æquilibrio.* [Vide Figure 63.]

SUPPOSE the Problem done, and parallel to *AD* draw *EF*, which ſhall be to *AE* as the Weight *E* to the Weight *D*. And from the Points *A* and *F* to the Line *BG* let fall the Perpendiculars *AB*, *FG*. Now ſince the Weights are, by Suppoſition, as the Lines *AE* and *EF*, expreſs thoſe Weights by thoſe Lines, the Weight *D* by the Line *EA*, and the Weight *E* by the Line *EF*. Therefore the Body, or *E*, directed by the Force of its own Weight, tends towards *F*. And by the oblique Force *EG* tends towards *G*. And the ſame Body *E* by the Weight *D* in the direct Force *AE*, is drawn towards *A*, and in the oblique Force *BE* is drawn towards *B*. Since therefore the Weights ſuſtain each other in Æquilibrio, the Force by which the Weight *E* is drawn towards *B*, ought to be equal to the contrary Force by which it tends towards *G*, that is, *BE* ought to be equal to *EG*. But now the Ratio of *AE* to *EF* is given by the Hypotheſis; and by reaſon of the given Angle *FEG*, there is alſo given the Ratio of *FE* to *EG*, to which *BE* is equal. Therefore there is given the Ratio of *AE* to *BE*. *AB* is alſo given in Length; and thence the Triangle *ABE*, and the Point *E* will eaſily be given. *Viz.* make $AB = a$, $BE = x$, and *AE* will be equal $\sqrt{aa+xx}$; moreover, let *AE* be to *BE* in the given Ratio of $d$ to $e$, and $e\sqrt{aa+xx}$ will $= dx$. And the Parts of the Æquation being ſquar'd and reduc'd, $eeaa = ddxx - eexx$, or $\frac{ea}{\sqrt{dd-ee}} = x$. Therefore the Length *BE* is found, which determines the Place of the Weight *E*. Q. E. F.

Now, if both Weights deſcend by oblique Lines given in Poſition, the Computation may be made thus. [*Vide Figure* 64.] Let $CD$ and $BE$ be oblique Lines given by Poſition, through which thoſe Weights deſcend. From the fix'd Tack $A$ to theſe Lines let fall the Perpendiculars $AC$, $AB$, and let the Lines $EG$, $DH$, erected from the Weights perpendicularly to the Horizon, meet them in the Points $G$ and $H$; and the Force by which the Weight $E$ endeavours to deſcend in a perpendicular Line, or the whole Gravity of $E$, will be to the Force by which the ſame Weight endeavours to deſcend in the oblique Line $BE$, as $GE$ to $BE$; and the Force by which it endeavours to deſcend in the oblique Line $BE$, will be to the Force by which it endeavours to deſcend in the Line $AE$, that is, to the Force by which the Thread $AE$ is diſtended [or ſtretch'd] as $BE$ to $AE$. And conſequently the Gravity of $E$ will be to the Tenſion of the Thread $AE$, as $GE$ to $AE$. And by the ſame Ratio the Gravity of $D$ will be to the Tenſion of the Thread $AD$, as $HD$ to $AD$. Let therefore the Length of the whole Thread $DA + AE$ be $c$, and let its Part $AE = x$, and its other Part $AD$ will $= c - x$. And becauſe $AEq - ABq$ is $= BEq$, and $ADq - ACq = CDq$; let, moreover, $AB = a$, and $AC = b$, and $BE$ will be $= \sqrt{xx - aa}$, and $CD = \sqrt{xx - 2cx + cc - bb}$. Moreover, ſince the Triangles $BEG$, $CDH$ are given in Specie, let $BE : EG :: f : E$, and $CD : DH :: f : g$, and $EG$ will $= \frac{E}{f}\sqrt{xx - aa}$, and $DH = \frac{g}{f}\sqrt{xx - 2cx + cc - bb}$. Wherefore ſince $GE : AE ::$ Weight $E$ : Tenſion of $AE$; and $HD : AD ::$ Weight $D$ : Tenſion of $AD$; and thoſe Tenſions are equal, you'll have $\frac{Ex}{\frac{E}{f}\sqrt{xx - aa}} =$ Tenſion of $AE =$ to the Tenſion $AD = \frac{Dc - Dx}{\frac{g}{f}\sqrt{xx - 2cx + cc - bb}}$; From the Reduction of which Æquation there comes out $gx\sqrt{xx - 2cx + cc - bb} = \overline{Dc - Dx}\sqrt{xx - aa}$, or

$$\begin{matrix} gg \\ -DD \end{matrix}\, x^4 \begin{matrix} -2ggc \\ +2DDc \end{matrix}\, x^3 \begin{matrix} +ggcc \\ -ggbb \\ -DDcc \\ +DDaa \end{matrix}\, xx - 2DDcaax + DDccaa = 0.$$

But if you deſire a Caſe wherein this Problem may be conſtructed by a Rule and Compaſs, make the Weight $D$ to the Weight $E$ as the Ratio $\frac{BE}{EG}$ to the Ratio $\frac{CD}{DH}$, and $g$ will become $= D$; and ſo in the Room of the precedent Æquation you'll have this, $-\frac{aa}{bb}xx - 2acx + aacc = 0$, or $x = \frac{ac}{a+b}$.

## PROBLEM XLIV.

*If on the String $DABCF$, that ſlides about the two Tacks $A$ and $B$, there are hung three Weights, $D$, $E$, $F$; $D$ and $F$ at the Ends of the String, and $E$ at its middle Point $C$, plac'd between the Tacks: From the given Weights and Poſition of the Tacks to find the Situation of the Point $C$, where the middle Weight hangs, and where they are in Æquilibrio.* [Vide Figure 65.]

SINCE the Tenſion of the Thread $AC$ is equal to the Tenſion of the Thread $AD$, and the Tenſion of the Thread $BC$ to the Tenſion of the Thread $BF$, the Tenſion of the Strings or Threads $AC$, $BC$, $EC$ will be as the Weights $D$, $E$, $F$. Then take the Parts of the Thread $CG$, $CH$, $CI$, in the ſame Ratio as the Weights. Compleat the Triangle $GHI$. Produce $IC$ till it meet $GH$ in $K$, and $GK$ will be $= KH$, and $CK = \frac{1}{2}CI$, and conſequently $C$ the Center of Gravity of the Triangle $GHI$. For, draw $PQ$ through $C$, perpendicular to $CE$, and perpendicular to that, from the Points $G$ and $H$, draw $GP$, $HQ$. And if the Force by which the Thread $AC$ by the Weight $D$ draws the Point $C$ towards $A$, be expreſs'd by the Line $GC$, the Force by which that Thread will draw the ſame Point towards $P$, will be expreſs'd by the Line $CP$; and the Force by which it draws it to $K$, will be expreſs'd by the Line $GP$. And in like Manner, the Forces by which the Thread $BC$, by Means of the Weight $F$, draws the ſame Point $C$

towards *B*, *Q*, and *K*, will be expreſs'd by the Lines *CH*, *CK*, and *HQ*; and the Force by which the Thread *CE*, by Means of the Weight *E*, draws that Point *C* towards *E*, will be expreſs'd by the Line *CI*. Now ſince the Point *C* is ſuſtain'd in Æquilibrio by equal Forces, the Sum of the Forces by which the Threads *AC* and *BC* do together draw *C* towards *K*, will be equal to the contrary Force by which the Thread *EC* draws that Point towards *E*; that is, the Sum $GP + HQ$ will be equal to $CI$; and the Force by which the Thread *AC* draws the Point *C* towards *P*, will be equal to the contrary Force by which the Thread *BC* draws the ſame Point *C* towards *Q*; that is, the Line *PC* is equal to the Line *CQ*. Wherefore, ſince *PG*, *CK*, and *QH* are Parallel, *GK* will be alſo $= KH$, and $CK\ (= \frac{GP + HQ}{2} = \frac{1}{2}CI.)$ Which was to be ſhewn. It remains therefore to determine the Triangle *GCK*, whoſe Sides *GC* and *HC* are given, together with the Line *CK*, which is drawn from the Vertex *C* to the middle of the Baſe. Let fall therefore from the Vertex *C* to the Baſe *CH* the Perpendicular *CL*, and $\frac{GCq - CHq}{2GH}$ will be $= KL = \frac{GCq - KCq - GKq}{2GK}$. For $2GK$ write $GH$, and having rejected the common Diviſor $GH$, and order'd the Terms, you'll have $GCq - 2KCq + CHq = 2GKq$, or $\sqrt{\frac{1}{2}GCq - KCq + \frac{1}{2}CHq} = GK$, having found *GK*, or *KH*, there are given together the Angles *GCK*, *KCH*, or *DAC*, *FBC*. Wherefore, from the Points *A* and *C* in theſe given Angles *DAC*, *FBC*, draw the Lines *AC*, *BC*, meeting in the Point *C*; and *C* will be the Point ſought.

But it is not always neceſſary to ſolve Queſtions that are of the ſame Kind, particularly by Algebra, but from the Solution of one of them you may moſt commonly infer the Solution of the other. As if now there ſhould be propos'd this Queſtion.

*The Thread ACDB being divided into the given Parts AC, CD, DB, and its Ends being faſten'd to the two Tacks given by Poſition, A and B; and if at the Points of Diviſion, C and D, there are hang'd the two Weights E and F; from the given Weight F, and the Situation of the Points C and D, to know the Weight E.* [Vide Figure 66.]

FROM the Solution of the former Problem the Solution of this may be eaſily enough found. Produce the Lines *AC*, *BD*, till they meet the Lines *DF*, *CE* in *G* and *H*; and the Weight *E* will be to the Weight *F*, as *DG* to *CH*.

And hence may appear a Method of making a Balance of only Threads, by which the Weight of any Body *E* may be known, from only one given Weight *F*.

## PROBLEM XLV.

*A Stone falling down into a Well, from the Sound of the Stone ſtriking the Bottom, to determine the Depth of the Well.*

LET the Depth of the Well be $x$, and if the Stone deſcends with an uniformly accelerated Motion through any given Space $a$, in any given Time $b$, and the Sound paſſes with an uniform Motion through the ſame given Space $a$, in the given Time $d$, the Stone will deſcend through the Space $x$ in the Time $b\sqrt{\frac{x}{a}}$; but the Sound which is cauſ'd by the Stone ſtriking upon the Bottom of the Well, will aſcend by the ſame Space $x$, in the Time $\frac{dx}{a}$. For the Spaces deſcrib'd by deſcending heavy Bodies, are as the Squares of the Times of Deſcent; or the Roots of the Spaces, that is, $\sqrt{x}$ and $\sqrt{a}$ are as the Times themſelves. And the Spaces $x$ and $a$, through which the Sound paſſes, are as the Times of Paſſage. And the Sum of theſe Times

$b\sqrt{\frac{x}{a}}$, and $\frac{dx}{a}$, is the Time of the Stone's falling to the Return of the Sound. This Time may be known by Obſervation. Let it be $t$, and you'll have $b\sqrt{\frac{x}{a}}+\frac{dx}{a}=t$. And $b\sqrt{\frac{x}{a}}=t-\frac{dx}{a}$. And the Parts being ſquar'd, $\frac{bbx}{a}=tt-\frac{2tdx}{a}+\frac{ddxx}{aa}$. And by Reduction $xx=\frac{2adt+abb}{dd}x-\frac{aatt}{dd}$. And having extracted the Root $x=\frac{adt+\frac{1}{2}abb}{dd}-\frac{ab}{2dd}\sqrt{bb+4dt}$.

## PROBLEM XLVI.

*Having given the Perimeter and Perpendicular of a right-angled Triangle, to find the Triangle.* [Vide Figure 67.]

LET $C$ be the right Angle of the Triangle, $ABC$ and $CD$ a Perpendicular let fall thence to the Baſe $AB$. Let there be given $AB+BC+AC=a$, and $CD=b$. Make the Baſe $AB=x$, and the Sum of the Sides will be $a-x$. Put $y$ for the Difference of the Legs, and the greater Leg $AC$ will be $=\frac{a-x+y}{2}$; the leſs $BC=\frac{a-x-y}{2}$. Now, from the Nature of a right-angled Triangle you have $ACq+BCq=ABq$, that is, $\frac{aa-2ax+xx+yy}{2}=xx$. And alſo $AB:AC::BC:DC$, therefore $AB\times DC=AC\times BC$, that is, $bx=\frac{aa-2ax+xx-yy}{4}$. By the former Æquation $yy$ is $=xx+2ax-aa$. By the latter $yy$ is $=xx-2ax+aa-4bx$. And conſequently $xx+2ax-aa=xx-2ax+aa-4bx$. And by Reduction $4ax+4bx=2aa$, or $x=\frac{aa}{2a+2b}$.

Geometrically thus. In every right-angled Triangle, as the Sum of the Perimeter and Perpendicular is to the Perimeter, ſo is half the Perimeter to the Baſe.

Subtract $2x$ from $a$, and there will remain $\frac{ab}{a+b}$, the Exceſs of the Sides above the Baſe. Whence again, as in every right-angled Triangle the Sum of the Perimeter and Perpendicular is to the Perimeter, ſo is the Perpendicular to the Exceſs of the Sides above the Baſe.

## PROBLEM XLVII.

### *Having given the Baſe AB of a right-angled Triangle, and the Sum of the Perpendicular, and the Legs CA + CB + CD; to find the Triangle.*

LET $CA+CB+CD=a$, $AB=b$, $CD=x$, and $AC+CB$ will be $=a-x$. Put $AC-CB=y$, and $AC$ will be $=\frac{a-x+y}{2}$, and $CB=\frac{a-x-y}{2}$. But $ACq+CBq$ is $=ABq$; that is, $\frac{aa-2ax+xx+yy}{2}=bb$. Moreover, $AC\times CB=AB\times CD$, that is, $\frac{aa-2ax+xx-yy}{4}=bx$. Which being compar'd, you have $2bb-aa+2ax-xx=yy=aa-2ax+xx-4bx$. And by Reduction, $xx=2ax+2bx-aa+bb$, and $x=a+b-\sqrt{2ab+2bb}$.

Geometrically thus. In any right-angled Triangle, from the Sum of the Legs and Perpendicular ſubtract the mean Proportional between the ſaid Sum and the double of the Baſe, and there will remain the Perpendicular.

*The ſame otherwiſe.*

Make $CA+CB+CD=a$, $AB=b$, and $AC=x$, and $BC$ will be $=\sqrt{bb-xx}$, $CD=\frac{x\sqrt{bb-xx}}{b}$. And $x+CB+CD=a$, or $CB+CD=a-x$. And therefore

$\frac{b+x}{b}\sqrt{bb-xx}=a-x$. And the Parts being ſquar'd and multiply'd by $bb$, there will be made $-x^4-2bx^3+2b^3x+b^4=aabb-2abbx+bbxx$. Which Æquation being order'd, by Tranſpoſition of Parts, after this Manner, $x^4+2bx^3\left\{\begin{matrix}+3bb\\+2ab\end{matrix}\right.xx\begin{matrix}+2b^3\\+2abb\end{matrix}x\left.\begin{matrix}+b^4\\+2ab^3\\+aabb\end{matrix}\right\}$ $=\begin{matrix}2bb\\2ab\end{matrix}xx\left.\begin{matrix}+4b^3\\+4abb\end{matrix}x\right\}\left\{\begin{matrix}+2b^4\\+2ab^3\end{matrix}\right\}$ and extracting the Roots on both Sides, there will ariſe $xx+bx+bb+ab=x+b\sqrt{2ab+2bb}$. And the Root being again extracted $x=-\frac{1}{2}b+\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}\pm\sqrt{b\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}-\frac{1}{4}bb-\frac{1}{2}ab}$.

*The Geometrical Conſtruction.* [Vide Figure 53.]

Take therefore $AB=\frac{1}{2}b$, $BC=\frac{1}{2}a$, $CD=\frac{1}{2}AB$, $AE$, a mean Proportional between $b$ and $AC$, and $EF=Ef$, a mean Proportional between $b$ and $DE$, and $BF$, $Bf$ will be the two Legs of the Triangle.

## PROBLEM XLVIII.

*Having given in the right-angled Triangle ABC, the Sum of the Sides AC + BC, and the Perpendicular CD, to find the Triangle.*

LET $AC+BC=a$, $CD=b$, $AC=x$, and $BC$ will $=a-x$, $AB=\sqrt{aa-2ax+2xx}$. Moreover, $CD:AC::BC:AB$. Therefore again $AB=\frac{ax-xx}{b}$. Wherefore $ax-xx=b\sqrt{aa-2ax+2xx}$; and the Parts being ſquar'd and order'd $x^4-2ax^3\begin{matrix}+aa\\-2bb\end{matrix}xx+2abbx-aabb=0$. Add to both Parts $aabb+b^4$, and there will be made $x^4-2ax^3\begin{matrix}+aa\\-2bb\end{matrix}xx+2abbx+b^4=aabb+b^4$. And the Root being extracted on both

Sides, $xx - ax - bb = -b\sqrt{aa+bb}$, and the Root being again extracted $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb - b\sqrt{aa+bb}}$.

### *The Geometrical Conſtruction.* [Vide Figure 69.]

Take $AB = BC = \frac{1}{2}a$. At $C$ erect the Perpendicular $CD = b$. Produce $DC$ to $E$, ſo that $DE$ ſhall $= DA$. And between $CD$ and $CE$ take a mean Proportional $CF$. And let a Circle, deſcrib'd from the Center $F$ and the Radius $BC$, cut the right Line $BC$ in $G$ and $H$, and $BG$ and $BH$ will be the two Sides of the Triangle.

### *The ſame otherwiſe.*

Let $AC + BC = a$, $AC - BC = y$, $AB = x$, and $DC = b$, and $\frac{a+y}{2}$ will $= AC$, $\frac{a-y}{2} = BC$, $\frac{aa+yy}{2} = ACq + BCq = ABq = xx$. $\frac{aa-yy}{4b} = \frac{AC \times BC}{DC} = AB = x$. Therefore $2xx - aa = yy = aa - 4bx$, and $xx = aa - 2bx$, and the Root being extracted $x = -b + \sqrt{bb+aa}$. Whence in the Conſtruction above $CE$ is the Hypothenuſe of the Triangle ſought. But the Baſe and Perpendicular, as well in this as the Problem above being given, the Triangle is thus expeditiouſly conſtructed. [*Vide Figure* 70.] Make a Parallelogram $CG$, whoſe Side $CE$ ſhall be the Baſis of the Triangle, and the other Side $CF$ the Perpendicular. And upon $CE$ deſcribe a Semicircle, cutting the oppoſite Side $PG$ in $H$. Draw $CH$, $EH$, and $CHE$ will be the Triangle ſought.

## PROBLEM XLIX.

*In a right-angled Triangle, having given the Sum of the Legs, and the Sum of the Perpendicular and Baſe, to find the Triangle.*

LET the Sum of the Legs $AC$ and $BC$ be [call'd] $a$, the Sum of the Baſe $AB$ and of the Perpendicular $CD$ be [call'd] $b$, let the Leg $AC = x$, the Baſe $AB = y$, and $BC$ will $= a - x$, $CD = b - y$, $aa - 2ax + 2xx = ACq + BCq$

$+BCq = ABq = yy$, $ax - xx = AC \times BC = AB \times CD = by - yy = by - aa + 2ax - 2xx$, and $by = aa - ax + xx$. Make its Square $a^4 - 2a^3x + 3aaxx - 2ax^3 + x^4$ equal to $yy \times bb$, that is, equal to $aabb - 2abbx + 2bbxx$. And ordering the Æquation, there will come out $x^4 - 2ax^3 \begin{matrix}+3aa\\-2bb\end{matrix} xx \begin{matrix}-2a^3\\+2abb\end{matrix} x \begin{matrix}+a^4\\-aabb\end{matrix} = 0$. Add to each Side of the Æquation $b^4 - aabb$, and there will come out $x^4 - 2ax^3 \begin{matrix}+3aa\\-2bb\end{matrix} xx \begin{matrix}-2a^3\\+2abb\end{matrix} x \begin{matrix}+a^4\\-2aabb\\+b^4\end{matrix} = b^4 - aabb$. And the Root being extracted on both Sides $xx - ax + aa - bb = -b\sqrt{bb - aa}$, and the Root being again extracted $x = \frac{1}{2}a \pm \sqrt{bb - \frac{3}{4}aa - b\sqrt{bb - aa}}$.

### *The Geometrical Construction.*

Take $R$ a mean Proportional between $b + a$ and $b - a$, and $S$ a mean Proportional between $R$ and $b - R$, and $T$ a mean Proportional between $\frac{1}{2}a + S$ and $\frac{1}{2}a - S$; and $\frac{1}{2}a + T$, and $\frac{1}{2}a - T$ will be the Sides of the Triangle.

## PROBLEM L.

*To subtend the given Angle CBD with the given right Line CD, so that if AD be drawn from the End of that right Line D to the Point A, given on the right Line CB produc'd, the Angle ADC shall be equal to the Angle ABD.* [Vide Figure 71.]

MAKE $CD = a$, $AB = b$, $BD = x$, and $BD$ will be $: BA :: CD : DA = \frac{ab}{x}$. Let fall the Perpendicular $DE$, and $BE$ will be $= \frac{BDq - ADq + BAq}{2BA} = \frac{xx - \frac{aabb}{xx} + bb}{2b}$. By reason of the given Triangle $DBA$, make $BD : BE :: b : e$, and you'll have again $BE = \frac{ex}{b}$, therefore $xx - \frac{aabb}{xx} + bb = 2ex$. And $x^4 - 2ex^3 + bbxx - aabb = 0$.

## PROBLEM LI.

### *Having the Sides of a Triangle given, to find the Angles.* [Vide Figure 72.]

LET the [given] Sides $AB = a$, $AC = b$, $BC = c$, to find the Angle $A$. Having let fall to $AB$ the Perpendicular $CD$, which is oppoſite to that Angle, you'll have in the firſt Place, $bb - cc = ACq - BCq = ADq - BDq = \overline{AD + BD} \times \overline{AD - BD} = AB \times \overline{2AD - AB} = 2AD \times a - aa$. And conſequently $\frac{1}{2}a + \frac{bb - cc}{2a} = AD$. Whence comes out this firſt *Theorem*. As $AB$ to $AC + BC$ ſo $AB - BC$ to a fourth Proportional $N$. $\frac{AB + N}{2} = AD$. As $AC$ to $AD$ ſo Radius to the Coſine of the Angle $A$.

Moreover, $DCq = ACq - ADq =$
$$\frac{2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4}{4aa} =$$
$$\frac{\overline{a + b + c} \times \overline{a + b - c} \times \overline{a - b + c} \times \overline{-a + b + c}}{4aa}.$$
Whence having multiply'd the Roots of the Numerator and Denominator by $b$, there is made this ſecond *Theorem*. As $2ab$ to a mean Proportional between $\overline{a + b + c} \times \overline{a + b - c}$ and $\overline{a - b + c} \times \overline{-a + b + c}$, ſo is Radius to the Sine of the Angle $a$.

Moreover, on $AB$ take $AE = AC$, and draw $CE$, and the Angle $ECD$ will be equal to half the Angle $A$. Take $AD$ from $AE$, and there will remain $DE = b - \frac{1}{2}a - \frac{bb - cc}{2a} = \frac{cc - aa + 2ab - bb}{2a} = \frac{\overline{c + a - b} \times \overline{c - a + b}}{2a}$. Whence $DEq = \frac{\overline{c + a - b} \times \overline{c + a - b} \times \overline{c - a + b} \times \overline{c - a + b}}{4aa}$. And hence is made the third and fourth *Theorem*, *viz.* As

$2ab$ to $\overline{c+a-b}\times\overline{c-a+b}$ (ſo $AC$ to $DE$) ſo Radius to the verſed Sine of the Angle $A$. And, as a mean Proportional between $a+b+c$, and $a+b-c$ to a mean Proportional between $c+a-b$, and $c-a+b$ (ſo $CD$ to $DE$) ſo Radius to the Tangent of half the Angle $A$, or the Cotangent of half the Angle to Radius.

Beſides, $CEq$ is $= CDq + DEq =$

$$\frac{2abb+bcc-baa-b^3}{a} = \frac{b}{a}\times\overline{c+a-b}\times\overline{c-a+b}.$$

Whence the fifth and ſixth *Theorem*. As a mean Proportional between $2a$ and $2b$ to a mean Proportional between $c+a-b$, and $c-a+b$, or as 1 to a mean Proportional between $\frac{c+a-b}{2a}$, and $\frac{c-a+b}{2b}$ (ſo $AC$ to $\frac{1}{2}CE$, or $CE$ to $DE$) ſo Radius to the Sine of $\frac{1}{2}$ the Angle $A$. And as a mean Proportional between $2a$ and $2b$ to a mean Proportional between $a+b+c$ and $a+b-c$ (ſo $CE$ to $CD$) ſo Radius to the Coſine of half the Angle $A$.

But if beſides the Angles, the Area of the Triangle be alſo ſought, multiply $CDq$ by $\frac{1}{4}ABq$, and the Root, *viz.* $\frac{1}{4}\sqrt{\overline{a+b+c}\times\overline{a+b-c}\times\overline{a-b+c}\times\overline{-a+b+c}}$ will be the Area ſought.

## PROBLEM LII.

### *From the Obſervation of four Places of a Comet, moving with an uniform right-lined Motion through the Heaven, to determine its Diſtance from the Earth, and Direction and Velocity of its Motion, according to the Copernican Hypotheſis.* [Vide Figure 73.]

IF from the Center of the Comet in the four Places obſerv'd, you let fall ſo many Perpendiculars to the Plane of the Ecliptick; and $A$, $B$, $C$, $D$, be the Points in that Plane on which the Perpendiculars fall; through thoſe Points draw the right Line $AD$, and this will be cut by the Perpendiculars in the ſame Ratio with the Line which the Comet deſcribes by its Motion; that is, ſo that $AB$ ſhall be to $AC$ as the Time between the firſt and ſecond Obſervation to the Time between the firſt and third; and $AB$ to $AD$ as the

Time between the firſt and ſecond to the Time between the firſt and fourth. From the Obſervations therefore there are given the Proportions of the Lines $AB$, $AC$, $AD$, to one another.

Moreover, let the Sun $S$ be in the ſame Plane of the Eclip-tick, and $EH$ an Arch of the Ecliptical Line in which the Earth moves; $E$, $F$, $G$, $H$, four Places of the Earth in the Times of the Obſervations, $E$ the firſt Place, $F$ the ſecond, $G$ the third, $H$ the fourth. Join $AE$, $BF$, $CG$, $DH$ and let them be produc'd till the three former cut the latter in $I$, $K$, and $L$, *viz.* $BF$ in $I$, $CG$ in $K$, $DH$ in $L$. And the Angles $AIB$, $AKC$, $ALD$ will be the Differences of the obſerv'd Longitudes of the Comet; $AIB$ the Difference of the Longitudes of the firſt and ſecond Place of the Comet; $AKC$ the Difference of the Longitudes of the firſt and third Place, and $ALD$ the Difference of the Longitudes of the firſt and fourth Place. There are given therefore from the Obſervations the Angles $AIB$, $AKC$, $ALD$.

Join $SE$, $SF$, $EF$; and by reaſon of the given Points $S$, $E$, $F$, and the given Angle $ESF$, there will be given the Angle $SEF$. There is given alſo the Angle $SEA$, as being the Difference of Longitude of the Comet and Sun in the Time of the firſt Obſervation. Wherefore, if you add its Complement to two right Angles, *viz.* the Angle $SEI$ to the Angle $SEF$, there will be given the Angle $IEF$. Therefore there are given the Angles of the Triangle $IEF$, together with the Side $EF$, and conſequently there is given the Side $IE$. And by a like Argument there are given $KE$ and $LE$. There are given therefore in Poſition the four Lines $AI$, $BI$, $CK$, $DL$, and conſequently the Problem comes to this, that four Lines being given in Poſition, we may find a fifth, which ſhall be cut by theſe four in a given Ratio.

Having let fall to $AI$ the Perpendiculars $BM$, $CN$, $DO$, by reaſon of the given Angle $AIB$ there is given the Ratio of $BM$ to $MI$. But $BM$ to $CN$ is in the given Ratio of $BA$ and $CA$, and by reaſon of the given Angle $CKN$ there is given the Ratio of $CN$ to $KN$. Wherefore, there is alſo given the Ratio of $BM$ to $KN$; and thence alſo the Ratio of $BM$ to $MI-KN$, that is, to $MN+IK$. Take $P$ to $IK$ as is $AB$ to $BC$, and ſince $MA$ is to $MN$ in the ſame Ratio, $P+MA$ will be to $IK+MN$ in the ſame Ratio, that is, in a given Ratio. Wherefore, there is given the Ratio of $BM$ to $P+MA$. And by a like Argument, if $Q$ be taken to $IL$ in the Ratio of $AB$ to $BD$, there will be given the Ratio of *BM* to *Q* + *MA*. And then the Ratio of *BM* to the Difference of *P* + *MA* and *Q* + *MA* will be given. But that Difference, *viz.* *P*—*Q*, or *Q*—*P* is given, and then there will be given *BM*. But *BM* being given, there are alſo given *P* + *MA* and *MI*, and thence, *MA*, *ME*, *AE*, and the Angle *EAB*.

Theſe being found, erect at *A* a Line perpendicular to the Plan of the Ecliptick, which ſhall be to the Line *EA* as the Tangent of the Comet's Latitude in the firſt Obſervation to Radius, and the End of that Perpendicular will be the Planet's Place in the firſt Obſervation. Whence the Diſtance of the Comet from the Earth is given in the Time of that Obſervation.

And after the ſame Manner, if from the Point *B* you erect a Perpendicular which ſhall be to the Line *BF* as the Tangent of the Comet's Latitude in the ſecond Obſervation to Radius, you'll have the Place of the Comet's Center in that ſecond Obſervation, and a Line drawn from the firſt Place to the ſecond, is that in which the Comet moves through the Heaven.

## PROBLEM LIII.

*If the given Angle CAD move about the angular Point A given in Poſition, and the given Angle CBD about the angular Point B given alſo in Poſition, on this Condition, that the Legs AD, BD, ſhall always cut one another in the right Line EF given likewiſe in Poſition; to find the Curve, which the Interſection C of the other Legs AC, BC, deſcribes.* [Vide Figure 74.]

PRODUCE *CA* to *d*, ſo that *Ad* ſhall be = *AD*, and produce *CB* to *δ*, ſo that *Bδ* ſhall be = to *BD*. Make the Angle *Ade* equal to the Angle *ADE*, and the Angle *Bδf* equal to the Angle *BDF*, and produce *AB* on both Sides till it meet *de* and *δf* in *e* and *f*. Produce alſo *ed* to *G*, that *dG* ſhall be = *δf*, and from the Point *C* to the Line *AB* draw *CH* parallel to *ed*, and *CK* parallel to *fδ*. And conceiving the Lines *eG*, *fδ* to remain immoveable while the Angles $CAD$, $CBD$, move by the aforeſaid Law about the Poles $A$ and $B$, $Gd$ will always be equal to $f\delta$, and the Triangle $CHK$ will be given in Specie. Make therefore $Ae = a$, $eG = b$, $Bf = c$, $AB = m$, $BK = x$, and $CK = y$. And $BK$ will be $: CK :: Bf : f\delta$. Therefore $f\delta = \frac{cy}{x} = Gd$. Take this from $Ge$, and there will remain $ed = b - \frac{cy}{x}$. Since the Triangle $CKH$ is given in Specie, make $CK : CH :: d : e$, and $CH : HK :: d : f$, and $CH$ will $= \frac{ey}{d}$, and $HK = \frac{fy}{d}$. And conſequently $AH = m - x - \frac{fy}{d}$. But $AH : HC :: Ae : ed$, that is, $m - x - \frac{f}{d}y : \frac{ey}{d} :: a : b - \frac{cy}{x}$. Therefore, by multiplying the Means and Extreams together, there will be made $mb - \frac{mcy}{x} - bx + cy - \frac{bf}{d}y + \frac{cfyy}{dx} = \frac{aey}{d}$. Multiply all the Terms by $dx$, and reduce them into Order, and there will come out $fcyy \begin{matrix} +dc \\ -ae \\ -fb \end{matrix} xy - dcmy - bdxx + bdmx = 0$. Where, ſince the unknown Quantities $x$ and $y$ aſcend only to two Dimenſions, it is evident, that the Curve Line that the Point $C$ deſcribes is a Conick Section. Make $\frac{ae + fb - dc}{c} = 2p$, and there will come out $yy = \frac{2pxy}{f} + \frac{dm}{f}y + \frac{bd}{fc}xx - \frac{bdm}{fc}x$. And the Square Root being extracted, $y = \frac{p}{f}x + \frac{dm}{2f} \pm$

$$\sqrt{\frac{pp}{ff}xx + \frac{bd}{fc}xx + \frac{pdm}{ff}x - \frac{bdm}{fc}x + \frac{ddmm}{4ff}}.$$

Whence we infer, that the Curve is an Hyperbola, if $\frac{bd}{fc}$ be Affirmative, or Negative and not greater than $\frac{pp}{ff}$; and a Parabola, if $\frac{bd}{fc}$ be Negative and equal to $\frac{pp}{ff}$; an Ellipſe or a Circle, if $\frac{bd}{fc}$ be both Negative and greater than $\frac{pp}{ff}$. Q.E.I.

## PROBLEM LIV.

### *To deſcribe a Parabola which ſhall paſs through four Points given.* [Vide Figure 75.]

LET thoſe given Points be *A*, *B*, *C*, *D*. Join *AB*, and biſect it in *E*. And through *E* draw *VE*, a right Line, which conceive to be the Diameter of a Parabola, the Point *V* being its Vertex. Join *AC*, and draw *DG* parallel to *AB*, and meeting *AC* in *G*. Make $AB = a$, $AC = b$, $AG = c$, $GD = d$. Upon *AC* take *AP* of any Length, and from *P* draw *PQ* parallel to *AB*, and conceiving *Q* to be a Point of the Parabola; make $AP = x$, $PQ = y$. And take any Æquation expreſſive of a Parabola, which determines the Relation between *AP* and *PQ*. As that *y* is $= e + fx \pm \sqrt{gg + hx}$.

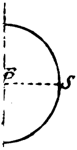
Now if *AP* or *x* be put $= 0$, the Point *P* falling upon *A*, *PQ* or *y* will be $= 0$, as alſo $= -AB$. And by writing in the aſſum'd Æquation 0 for *x*, you'll have $y = e \pm \sqrt{gg}$, that is, $= e \pm g$. The greater of which Values of *y*, $e + g$ is $= 0$, the leſſer $e - g = -AB$, or to $-a$. Therefore $e = -g$, and $e - g$, that is, $-2g = -a$, or $g = \frac{1}{2}a$. And ſo in room of the aſſum'd Æquation you'll have this $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + hx}$.

Moreover, if *AP* or *x* be made $= AC$, ſo that the Point *P* falls upon *C*, you'll have again $PQ = 0$. For *x* therefore in the laſt Æquation write *AC* or *b*, and for *y* write 0; and you'll have $0 = -\frac{1}{2}a + fb + \sqrt{\frac{1}{4}aa + hb}$, or $\frac{1}{2}a - fb = \sqrt{\frac{1}{4}aa + hb}$; and the Parts being ſquar'd $-afb + ffbb = hb$, or $ffb - fa = h$. And ſo, in room of the aſſum'd Æquation, there will be had this, $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + ffbx - fax}$.

Moreover, if *AP* or *x* be made $= AG$ or *c*, *PQ* or *y* will be $= -GD$ or $-d$. Wherefore, for *x* and *y* in the laſt Æquation write *c* and $-d$, and you'll have $-d = -$

$\frac{1}{2}a$

60



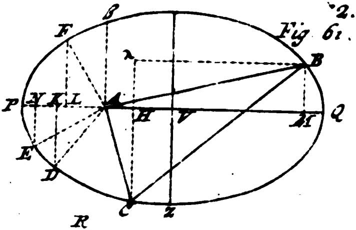


Fig 61.

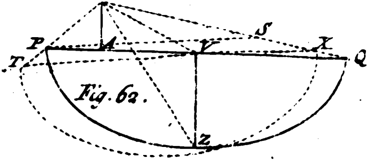


Fig. 62.

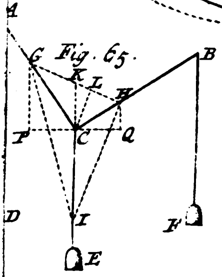


Fig. 65.

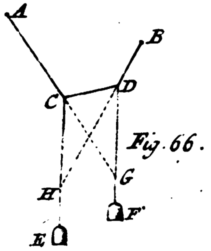


Fig. 66.

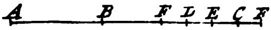


Fig. 55.

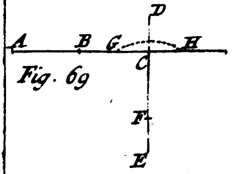


Fig. 69

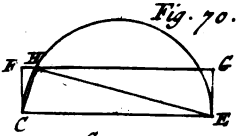


Fig. 70.

. 71 .

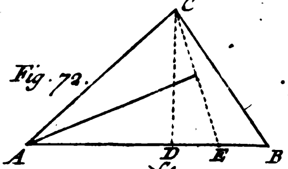
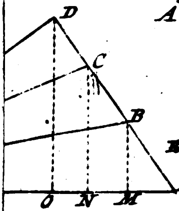


Fig. 72.



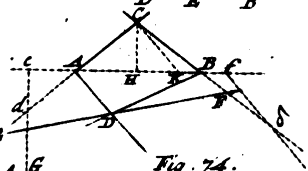


Fig. 74.

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$\frac{1}{2}a + fc = \sqrt{\frac{1}{4}aa + ffbc - fac}$, or $\frac{1}{2}a - d - fc = \sqrt{\frac{1}{4}aa + ffbc - fac}$. And the Parts being ſquar'd, $-ad - fac + dd + 2dcf + ccff = ffbc - fac$. And the Æquation being order'd and reduc'd, $ff = \frac{2d}{b-c}f + \frac{dd-ad}{bc-cc}$. For $b-c$, that is, for $GC$ write $k$, and that Æquation will become $ff = \frac{2d}{k}f + \frac{dd-ad}{kc}$. And the Root being extracted, $f = \frac{d}{k} \pm \sqrt{\frac{ddc + ddk - adk}{kkc}}$. But $f$ being found, the Parabolick Æquation, *viz.* $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + ffbx - fax}$ will be fully determin'd; by whoſe Conſtruction therefore the Parabola will alſo be determin'd. The Conſtruction is thus: Draw $CH$ parallel to $BD$ meeting $DG$ in $H$. Between $DG$ and $DH$ take a mean Proportional $DK$, and draw $EI$ parallel to $CK$, biſecting $AB$ in $E$, and meeting $DG$ in $I$. Then produce $IE$ to $V$, ſo that $EV$ ſhall be to $EI :: EBq : DIq - EBq$, and $V$ will be the Vertex, $VE$ the Diameter, and $\frac{BEq}{VE}$ the *Latus Rectum* of the Parabola ſought.

## PROBLEM LV.

### *To deſcribe a Conick Section through five Points given.* [Vide Figure 76.]

LET thoſe Points be $A$, $B$, $C$, $D$, $E$. Join $AC$, $BE$, cutting one another in $H$. Draw $DI$ parallel to $BE$, and meeting $AC$ in $I$. As alſo $EK$ parallel to $AC$, and meeting $DI$ produc'd in $K$. Produce $ID$ to $F$, and $EK$ to $G$; ſo that $AHC$ ſhall be : $BHE :: AIC : FID :: EKG : FKD$, and the Points $F$ and $G$ will be in a Conick Section; as is known.

But you ought to obſerve this, if the Point $H$ falls between all the Points $A$, $C$, and $B$, $E$, or without them all, the Point $I$ muſt either fall between all the Points $A$, $C$, and $F$, $D$, or without them all; and the Point $K$ between all the Points $D$, $F$, and $E$, $G$, or without them all. But if the Point $H$ falls between the two Points $A$, $C$, and without the other two $B$, $E$, or between thoſe two $BE$, and without the other two $AC$, the Point $I$ ought to fall between two of the Points $A\ C$ and $F, D$, and without the other two of them; and in like Manner, the Point $K$ ought to fall between two of the Points $D, F$, and $E, G$, and without Side of the two other of them; which will be done by taking $IF$, $KG$, on this or that Side of the Points $I$, $K$, according to the Exigency of the Problem. Having found the Points $F$ and $G$, biſect $AC$ and $EG$ in $N$ and $O$; alſo $BE$, $FD$ in $L$ and $M$. Join $NO$, $LM$, cutting one another in $R$; and $LM$ and $NO$ will be the Diameters of the Conick Section, $R$ its Center, and $BL$, $FM$, Ordinates to the Diameter $LM$. Produce $LM$ on both Sides, if there be Occaſion, to $P$ and $Q$, ſo that $BLq$ ſhall be to $FMq :: PLQ : PMQ$, and $P$ and $Q$ will be the Vertex's of the Conick Section, and $PQ$ the *Latus Tranſverſum*. Make $PLQ : LBq :: PQ : T$, and $T$ will be the *Latus Rectum*. Which being known, the Figure is known.

It remains only that we may ſhew how $LM$ is to be produc'd each Way to $P$ and $Q$, ſo that $BLq$ may be : $FMq :: PLQ : PMQ$, *viz.* $PLQ$, or $PL \times LQ$, is $\overline{PR - LR} \times \overline{PR + LR}$; for $PL$ is $PR - LR$, and $LQ$ is $RQ + LR$, or $PR + LR$. Moreover, $\overline{PR - LR} \times \overline{PR + LR}$, by multiplying, becomes $PRq - LRq$. And after the ſame Manner, $PMq$ is $\overline{PR + RM} \times \overline{PR - RM}$, or $PRq - RMq$. Therefore $BLq : FMq :: PRq - LRq : PRq - RMq$; and by dividing, $BLq - FMq : FMq :: RMq - LRq : PRq - RMq$. Wherefore ſince there are given $BLq - FMq$, $FMq$ and $RMq - LRq$, there will be given $PRq - RMq$. Add the given Quantity $RMq$, and there will be given the Sum $PRq$, and conſequently its Root $PR$, to which $QR$ is equal.

## PROBLEM LVI.

### *To deſcribe a Conick Section which ſhall paſs through four given Points, and in one of thoſe Points ſhall touch a right Line given in Poſition.* [Vide Figure 77.]

LET the four given Points be $A, B, C, D$, and the right Line given in Poſition be $AE$, which let the Conick Section touch in the Point $A$. Join any two Points $D, C$, and let $DC$ produc'd, if there be Occaſion for it, meet the Tangent in $E$. Through the fourth Point $B$ draw $BF$ parallel to $DC$, which ſhall meet the ſame Tangent in $F$. Alſo draw $DI$ parallel to the Tangent, and which may meet $BF$ in $I$. Upon $FB$, $DI$, produc'd, if there be Occaſion, take $FG$, $HI$, of ſuch Length as $AEq : CED :: AFq : BFG :: DIH : BIG$. And the Points $G$ and $H$ will be in a Conick Section as is known; if you only take $FG$, $IH$, on the right Sides of the Points $F$ and $I$, according to the Rule deliver'd in the former Problem. Biſect $BG$, $DC$, $DH$, in $K, L$, and $M$. Join $KL$, $AM$, cutting one another in $O$, and $O$ will be the Center, $A$ the Vertex, and $HM$ an Ordinate to the Semi-Diameter $AO$; which being known, the Figure is known.

## PROBLEM LVII.

### *To deſcribe a Conick Section which ſhall paſs through three given Points, and touch right Lines given in Poſition in two of thoſe Points.* [Vide Figure 78.]

LET thoſe given Points be $A, B, C$, touching $AD$, $BD$, in the Points $A$ and $B$, and let $D$ be the common Interſection of thoſe Tangents. Biſect $AB$ in $E$. Draw $DE$, and produce it till in $F$ it meets $CF$ drawn parallel to $AB$; and $DF$ will be the Diameter, and $AE$ and $CF$ the Ordinates to [that] Diameter. Produce $DF$ to $O$, and on $DO$ take $OV$ a mean Proportional between $DO$ and $EO$, on this Condition, that alſo $AEq : CFq :: VE \times \overline{VO + OE} : VF$

$: VF \times \overline{VO + OF}$; and $V$ will be the Vertex, and $O$ the Center of the Figure. Which being known, the Figure will alſo be known. But $VE$ is $= VO - OE$, and conſequently $VE \times \overline{VO + OE} = \overline{VO - OE} \times \overline{VO + OE} = VOq - OEq$. Beſides, becauſe $VO$ is a mean Proportional between $DO$ and $EO$, $VOq$ will be $= DOE$, and conſequently $VOq - OEq = DOE - OEq = DEO$. And by a like Argument you'll have $VF \times \overline{VO + OF} = VOq - OFq = DOE - OFq$. Therefore $AEq : CFq :: DEO : DOE - OFq$. $OFq$ is $= EOq - 2FEO + FEq$. And conſequently $DOE - OFq = DOE - OEq + 2FEO - FEq = DEO + 2FEO - FEq$. And $AEq : CFq :: DEO : DEO + 2FEO - FEq :: DE : DE + 2FE - \frac{FEq}{EO}$. Therefore there is given $DE + 2FE - \frac{FEq}{EO}$. Take away from this given Quantity $DE + 2FE$, and there will remain $\frac{FEq}{EO}$ given. Call that $N$; and $\frac{FEq}{N}$ will be $= EO$, and conſequently $EO$ will be given. But $EO$ being given, there is alſo given $VO$, the mean Proportional between $DO$ and $EO$.

After this Way, by ſome of *Apollonius*'s Theorems, theſe Problems are expeditiouſly enough ſolv'd; which yet may be ſolv'd by Algebra without thoſe Theorems. As if the firſt of the three laſt Problems be propos'd: [*Vide Figure* 78.] Let the five given Points be $A$, $B$, $C$, $D$, $E$, through which the Conick Section is to paſs. Join any two of them, $A$, $C$, and any other two, $B$, $E$, by Lines cutting (or interſecting) one another in $H$. Draw $DI$ parallel to $BE$ meeting $AC$ in $I$; as alſo any other right Line $KL$ meeting $AC$ in $K$, and the Conick Section in $L$. And imagine the Conick Section to be given, ſo that the Point $K$ being known, there will at the ſame Time be known the Point $L$; and making $AK = x$, and $KL = y$, to expreſs the Relation between $x$ and $y$, aſſume any Æquation which generally expreſſes the Conick Sections; ſuppoſe this, $a + bx + cxx + dy + exy + yy = 0$. Wherein $a$, $b$, $c$, $d$, $e$, denote determinate Quantities with their Signs, but $x$ and $y$ indeterminate Quantities. Now if we can find the determinate Quantities $a$, $b$, $c$, $d$, $e$, the Conick Section will be known. If therefore the Point $L$ falls upon the Point $A$, in that Caſe $AK$ and $KL$, that is, $x$ and $y$, will be $0$. Then all the Terms of the

Æquation beſides $a$ will vaniſh, and there will remain $a = 0$. Wherefore $a$ is to be blotted out in that Æquation, and the other Terms $bx + cxx + dy + exy + yy$ will be $= 0$. But if $L$ falls upon $C$, $AK$, or $x$, will be $= AC$, and $LK$ or $y$, $= 0$. Put therefore $AC = f$, and by ſubſtituting $f$ for $x$ and $0$ for $y$, the Æquation for the Curve $bx + cxx + dy + exy + yy = 0$, will become $bf + cff = 0$, or $b = -cf$. And having writ in that Æquation $-cf$ for $b$, it will become $-cfx + cxx + dy + exy + yy = 0$. Moreover, if the Point $L$ falls upon the Point $B$, $AK$ or $x$ will be $= AH$, and $KL$ or $y = BH$. Put therefore $AH = g$, and $BH = h$, and then write $g$ for $x$ and $h$ for $y$, and the Æquation $-cfx + cxx$, &c. will become $-cfg + cgg + dh + egh + hh = 0$. Now if the Point $L$ falls upon $E$, $AK$ will be $= AH$, or $x = g$, and $KL$ or $y = HE$. For $HE$ therefore write $-k$, with a Negative Sign, becauſe $HE$ lies on the contrary Side of the Line $AC$, and by ſubſtituting $g$ for $x$ and $-k$ for $y$, the Æquation $-cfx + cxx$, &c. will become $-cfx + cgg - dk - egk + kk = 0$. Take away this from the former Æquation $-cfg + cgg + dh + egh + hh$, and there will remain $dh + egh + hh + dk + egk - kk = 0$. Divide this by $h + k$, and there will come out $d + eg + h - k = 0$. Take away this multiply'd by $h$ from $-cfg + cgg + dh + egh + hh = 0$, and there will remain $-cfg + cgg + hk = 0$, or $\frac{hk}{-gg + fg} = c$. Laſtly, if the Point $L$ falls upon the Point $D$, $AK$ or $x$ will be $= AI$, and $KL$ or $y$ will be $= ID$. Wherefore, for $AI$ write $m$, and for $ID$, $n$, and likewiſe for $x$ and $y$ ſubſtitute $m$ and $n$, and the Æquation $-cfx + cxx$, &c. will become $-cfm + cmm + dn + emn + nn = 0$. Divide this by $n$, and there will come out $\frac{-cfm + cmm}{n} + d + em + n = 0$. Take away $d + eg + h - k = 0$, and there will remain $\frac{-cfm + cmm}{n} + em - eg + n - h + k = 0$, or $\frac{cmm - cfm}{n} + n - h + k = eg - em$. But now by reaſon of the given Points $A$, $B$, $C$, $D$, $E$, there are given $AC$, $AH$, $AI$, $BH$, $EH$, $DI$, that is, $f$, $g$, $m$, $h$, $k$, $n$. And conſequently by the Æquation $\frac{hk}{fg-gg}=c$ there is given *C*. But *c* being given by the Æquation $\frac{cmm-cfm}{n}+n-h+k=eg-em$ there is given $eg-em$. Divide this given Quantity by the given one $g-m$, and there will come out the given *e*. Which being found, the Æquation $d+eg+h-k=0$, or $d=k-h-eg$, will give *d*. And theſe being known, there will at the ſame Time be determin'd the Æquation expreſſive of the Conick Section ſought, *viz.* $cfx=cxx+dy+exy+yy$. And from that Æquation, by the Method of *Des Cartes*, the Conick Section will be determin'd.

Now if the four Points *A*, *B*, *C*, *E*, and the Poſition of the right Line *AF*, which touches the Conick Section in one of thoſe Points, *A* were given, the Conick Section may be thus more eaſily determin'd. Having found, as above, the Æquations $cfx=cxx+dy+exy+yy$, $d=k-h-eg$, and $c=\frac{hk}{fg-gg}$, conceive the Tangent *AF* to meet the right Line *EH* in *F*, and then the Point *L* to be moved along the Perimeter of the Figure *CDE* till it fall upon the Point *A*; and the ultimate Ratio of *LK* to *AK* will be the Ratio of *FH* to *AH*, as will be evident to any one that contemplates the Figure. Make $FH=p$, and in this Caſe where *LK*, *AK*, are in a vaniſhing State, you'll have $p:g::y:x$, or $\frac{gy}{p}=x$. Wherefore for *x*, in the Æquation $cfx=cxx+dy+exy+yy$, write $\frac{gy}{p}$, and there will ariſe $\frac{cfgy}{p}=\frac{cggyy}{pp}+dy+\frac{egyy}{p}+yy$. Divide all by *y*, and there will come out $\frac{cfg}{p}=\frac{cggy}{pp}+d+\frac{egy}{p}+y$. Now becauſe the Point *L* is ſuppos'd to fall upon the Point *A*, and conſequently *KL*, or *y*, to be infinitely ſmall or nothing, blot out the Terms which are multiply'd by *y*, and there will remain $\frac{cfg}{p}=d$. Wherefore make $\frac{hk}{fg-gg}=c$, then $\frac{cfg}{p}=d$. Laſtly, $\frac{k-h-d}{g}=e$, and having found $c$, $d$, and $e$, the Æquation $cfx = cxx + dy + exy + yy$ will determine the Conick Section.

If, laſtly, there are only given the three Points $A$, $B$, $C$, together with the Poſition of the two right Lines $AT$, $CT$, which touch the Conick Section in two of thoſe Points, $A$ and $C$, there will be obtain'd, as above, this Æquation expreſſive of a Conick Section, $cfx = cxx + dy + exy + yy$. [*Vide Figure* 80.]. Then if you ſuppoſe the Ordinate $KL$ to be parallel to the Tangent $AT$, and it be conceiv'd to be produc'd, till it again meets the Conick Section in $M$, and that Line $LM$ to approach to the Tangent $AT$ till it coincides with it at $A$, the ultimate [or evaneſcent] Ratio of the Lines $KL$ and $KM$ to one another, will be a Ratio of Æquality, as will appear to any one that contemplates the Figure. Wherefore in that Caſe $KL$ and $KM$ being equal to each other, that is, the two Values of $y$, (*viz.* the Affirmative one $KL$, and the Negative one $KM$) being equal, thoſe Terms of the Æquation ($cfx = cxx + dy + exy + yy$) in which $y$ is of an odd Dimenſion, that is, the Terms $dy + exy$ in reſpect of the Term $yy$, wherein $y$ is of an even Dimenſion, will vaniſh. For otherwiſe the two Values of $y$, *viz.* the Affirmative and the Negative, cannot be equal; and in that Caſe $AK$ is infinitely leſs than $LK$, that is $x$ than $y$, and conſequently the Term $exy$ than the Term $yy$. And conſequently being infinitely leſs, may be reckon'd for nothing. But the Term $dy$, in reſpect of the Term $yy$, will not vaniſh as it ought to do, but will grow ſo much the greater, unleſs $d$ be ſuppos'd to be nothing. Therefore the Term $dy$ is to be blotted out, and ſo there will remain $cfx = cxx + exy + yy$, an Æquation expreſſive of a Conick Section. Conceive now the Tangents $AT$, $CT$, to meet one another in $T$, and the Point $L$ to come to approach to the Point $C$, till it coincides with it. And the ultimate Ratio of $KL$ to $KC$ will be that of $AT$ to $AC$. $KL$ was $y$; $AK$, $x$; and $AC$, $f$; and conſequently $KC$, $f - x$; make $AT = g$, and the ultimate Ratio of $y$ to $f - x$, will be the ſame as of $g$ to $f$. The Æquation $cfx = cxx + exy + yy$, ſubtracting on both Sides $cxx$, becomes $cfx - cxx = exy + yy$, that is, $\overline{f - x}$ into $cx = y$ into $\overline{ex + y}$. Therefore $y : f - x :: cx : ex + y$, and conſequently $g : f :: cx : ex + y$. But the Point $L$ falling upon $C$, $y$ becomes nothing. Therefore $g : f :: cx : ex$. Divide the latter Ratio by $x$, and it will become $g : f :: c : e$, and $\frac{cf}{g}=e$. Wherefore, if in the Æquation $cfx=cxx+exy+yy$, you write $\frac{cf}{g}$ for $e$, it will become $cfx=cxx+\frac{cf}{g}xy+yy$, an Æquation expreſſive of a Conick Section. Laſtly, draw $BH$ parallel to $KL$, or $AT$, from the given Point $B$, through which the Conick Section ought to paſs, and which ſhall meet $AC$ in $H$, and conceiving $KL$ to come towards $BH$, till it coincides with it, in that Caſe $AH$ will be $=x$, and $BH=y$. Call therefore the given $AH=m$, and the given $BH=n$, and then for $x$ and $y$, in the Æquation $cfx=cxx+\frac{cf}{g}xy+yy$, write $m$ and $n$, and there will ariſe $cfm=cmm+\frac{cf}{g}mn+nn$. Take away on both Sides $cmm+\frac{cf}{g}mn$, and there will come out $cfm-cmm-\frac{cf}{g}mn=nn$. Put $f-m-\frac{fn}{g}=s$, and $csm$ will be $=nn$. Divide each Part of the Æquation by $sm$, and there will ariſe $c=\frac{nn}{sm}$. But having found $c$, the Æquation for the Conick Section is determin'd ($cfx=cxx+\frac{cf}{g}xy+yy$). And then, by the Method of *Des Cartes*, the Conick Section is given, and may be deſcrib'd.

PRO-

## PROBLEM LVIII.

*Having given the Globe A, and the Poſition of the Wall DE, and BD the Diſtance of the Center of the Globe B from the Wall; to find the Bulk of the Globe B, on this Condition, that if the Globe A, (whoſe Center is in the Line BD, which is perpendicular to the Wall, and produc'd out beyond B) be moved in free abſolute Space, and where Gravity can't act, with an uniform Motion towards D, till it falls upon [or ſtrikes againſt] the other quieſcent Globe B; and that Globe B, after it is reflected from the Wall, ſhall meet the Globe A in the given Point C.* [Vide Figure 81.]

LET the Velocity of the Globe *A* before Reflection be *a*, and by *Problem* 12. the Velocity of the Globe *A* will be after Reflection $= \frac{aA - aB}{A + B}$, and the Velocity of the Globe *B* after Reflection will be $= \frac{2aA}{A + B}$. Therefore the Velocity of the Globe *A* to the Velocity of the Globe *B* is as $A - B$ to $2A$. On $GD$ take $gD = GH$, *viz.* to the Diameter of the Globe *B*, and thoſe Velocities will be as $GC$ to $Gg + gC$. For when the Globe *A* ſtruck upon the Globe *B*, the Point *G*, which being on the Surface of the Globe *B* is moved in the Line *AD*, will go through the Space $Gg$ before that Globe *B* ſhall ſtrike againſt the Wall, and through the Space $gC$ after it is reflected from the Wall; that is, through the whole Space $Gg + gC$, in the ſame Time wherein the Point *F* of the Globe *A* ſhall paſs through the Space *GC*, ſo that both Globes may again meet and ſtrike one another in the given Point *C*. Wherefore, ſince the Intervals *BC* and *CD* are given, make $BC = m$, $BD + CD = n$, and $BG = x$, and $GC$ will be $= m + x$, and $Gg + gC = GD + DC - 2gD = GB + BD + DC - 2GH = x + n - 4x$, or $= n - 3x$. Above you had $A - B$ to $2A$, as the Velocity of the Globe *A* to the Velocity of the

Globe *B*, and the Velocity of the Globe *A* to the Velocity of the Globe *B*, as $GC$ to $Gg+gC$, and conſequently $A-B$ to $2A$, as $GC$ to $Gg+gC$; therefore ſince $GC$ is $=m+x$, and $Gg+gC=n-3x$, $A-B$ will be to $2A$ as $m+x$ to $n-3x$. Moreover, the Globe *A* is to the Globe *B* as the Cube of its Radius *AF* to the Cube of the others Radius *GB*; that is, if you make the Radius *AF* to be *s*, as $s^3$ to $x^3$; therefore $s^3-x^3:2s^3$ $(::A-B:2A)::m+x:n-3x$. And multiplying the Means and Extreams by one another, you'll have this Æquation, $s^3n-3s^3x-nx^3+3x^4=2ms^3+2sx^3$. And by Reduction $3x^4-nx^3-5s^3x\begin{matrix}+s^3n\\-2s^3m\end{matrix}=0$. From the Conſtruction of which Æquation there will be given *x*, the Semi-Diameter of the Globe *B*; which being given, that Globe is alſo given. Q. E. F.

But note, when the Point *C* lies on contrary Sides of the Globe *B*, the Sign of the Quantity $2m$ muſt be chang'd, and written $3x^4-nx^3-5s^3x\begin{matrix}+s^3n\\+2s^3m\end{matrix}=0$.

If the Globe *B* were given, and the Globe *A* ſought on this Condition, that the two Globes, after Reflection, ſhould meet in *C*, the Queſtion would be eaſier; *viz.* in the laſt Æquation found, *x* would be ſuppos'd to be given, and *s* to be ſought. Whereby, by a due Reduction of that Æquation, the Terms $-5s^3x+s^3n-2s^3m$ being tranſlated to the contrary Side of the Æquation, and each Part divided by $5x-n+2m$, there would come out $\frac{3x^4-nx^3}{5x-n+2m}=s^3$. Where *s* will be obtain'd by the bare Extraction of the Cube Root.

Now if both Globes being given, you were to find the Point *C*, in which both would fall upon one another after Reflection, the ſame Æquation by due Reduction would give $m=\frac{1}{2}n-\frac{5}{2}x+\frac{3x^4-x^3n}{2s^3}$; that is, $BC=\frac{1}{2}Hg+\frac{1}{2}gC-\frac{B}{2A}\times\overline{HD+DC}$. For above, $n-3x$ was $=Gg+gC$. Whence, if you take away $2x$, or *GH*, there will remain $n-5x=Hg+gC$. The Half whereof is $\frac{1}{2}n-\frac{5}{2}x=\frac{1}{2}Hg+\frac{1}{2}gC$. Moreover, from *n*, or $BD+CD$, take away *x*, or *BH*, and there will remain $n-x$, or

*HD*

$\overline{HD+CD}$. Whence, ſince $\frac{x^3}{2s^3} = \frac{B}{2A}$, you'll have $\frac{x^3}{2s^3} \times \overline{n-x}$, or $\frac{nx^3 - x^4}{2s^3} = \frac{B}{2A} \times \overline{HD+CD}$. And the Signs being chang'd, $\frac{x^4 - nx^3}{2s^3} = -\frac{B}{2A} \times \overline{HD+CD}$.

## PROBLEM LIX.

*If two Globes, A and B, are join'd together by a ſmall Thread PQ, and the Globe B hanging on the Globe A; if you let fall the Globe A, ſo that both Globes may begin to fall together by the ſole Force of Gravity in the ſame perpendicular Line PQ; and then the lower Globe B, after it is reflected upwards from the Bottom or Horizontal Plane FG, it ſhall meet the upper Globe A, as falling, in a certain Point D; from the given Length of the Thread PQ, and the Diſtance DF of that Point D from the Bottom, to find the Height PF, from which the upper Globe A ought to be let fall to [cauſe] this Effect.* [Vide Figure 83.]

LET $a$ be the Length of the Thread $PQ$. In the Perpendicular $PQRF$, from $F$ upwards take $FE$ equal to $QR$ the Diameter of the lower Globe, ſo that when the loweſt Point $R$ of that Globe falls upon the Bottom in $F$, its upper Point $Q$ ſhall poſſeſs the Place $E$; and let $ED$ be the Diſtance through which that Globe, after it is reflected from the Bottom, ſhall, by aſcending, paſs, before it meets the upper falling Globe in the Point $D$. Therefore, by reaſon of the given Diſtance $DF$ of the Point $D$ from the Bottom, and the Diameter $EF$ of the inferiour Globe, there will be given their Difference $DE$. Let that $= b$, and let the Depth $RF$, or $QE$, through which that lower Globe by falling before it touches the Bottom be $= x$, if it be unknown. And having found $x$, if to it you add $EF$ and

$PQ$; there will be had the Height $PF$, from which the upper Globe ought to fall to have the deſir'd Effect.

Since therefore $PQ$ is $=a$, and $QE=x$, $PE$ will be $=a+x$. Take away $DE$ or $b$, and there will remain $PD=a+x-b$. But the Time of the Deſcent of the Globe $A$ is as the Root of the Space deſcrib'd in falling, or $\sqrt{a+x-b}$, and the Time of the Deſcent of the other Globe $B$ as the Root of the Space deſcrib'd by [its] falling, or $\sqrt{x}$, and the Time of its Aſcent as the Difference of that Root, and of the Root of the Space which it would deſcribe by falling only from $Q$ to $D$. For this Difference is as the Time of Deſcent from $D$ to $E$, which is equal to the Time of Aſcent from $E$ to $D$. But that Difference is $\sqrt{x}-\sqrt{x-b}$. Whence the Time of Deſcent and Aſcent together will be as $2\sqrt{x}-\sqrt{x-b}$. Wherefore, ſince this Time is equal to the Time of Deſcent of the upper Globe, the $\sqrt{a+x-b}$ will be $=2\sqrt{x}-\sqrt{x-b}$. The Parts of which Æquation being ſquar'd, you'll have $a+x-b=5x-b-4\sqrt{xx-bx}$, or $a=4x-4\sqrt{xx-bx}$; and the Æquation being order'd, $4x-a=4\sqrt{xx-bx}$; and ſquaring the Parts of that Æquation again, there ariſes $16xx-8ax+aa=16xx-16bx$, or $aa=8ax-16bx$; and dividing all by $8a-16b$, you'll have $\frac{aa}{8a-16b}=x$. Make therefore as $8a-16b$ to $a$, ſo $a$ to $x$, and you'll have $x$ or $QE$. Q. E. I.

Now if from the given $QE$ you are to find the Length of the Thread $PQ$ or $a$; the ſame Æquation $aa=8ax-16bx$, by extracting the affected Quadratick Root, would give $a=4x-\sqrt{16xx-16bx}$; that is, if you take $QY$ a mean Proportional between $QD$ and $QE$, $PQ$ will be $=4EY$. For that mean Proportional will be $\sqrt{x\times\overline{x-b}}$, or $\sqrt{xx-bx}$; which ſubtracted from $x$, or $QE$, leaves $EY$, the Quadruple whereof is $4x-4\sqrt{xx-bx}$.

But if from the given Quantities $QE$, or $x$, as alſo the Length of the Thread $PQ$, or $a$, there were ſought the Point $D$ on which the upper Globe falls upon the under one; the Diſtance $DE$, or $b$, of that Point from the given Point $E$, will be had from the precedent Æquation $aa=8ax-16bx$ by transferring $aa$ and $16bx$ to the contrary Sides of the

Æquation with the Signs chang'd, and by dividing the whole by $16x$. There will ariſe $\frac{8ax-aa}{16x}=b$. Make therefore as $16x$ to $8x-a$, ſo $a$ to $b$, and you'll have $b$ or $DE$.

Hitherto I have ſuppos'd the Globes ty'd together by a ſmall Thread to be let fall together. Which, if they are let fall at different Times connected by no Thread, ſo that the upper Globe $A$, for Example, being let fall firſt, ſhall deſcend through the Space $PT$ before the other Globe begins to fall, and from the given Diſtances $PT$, $PQ$, and $DE$, you are to find the Height $PF$, from which the upper Globe ought to be let fall, ſo that it ſhall fall upon the inferior or lower one at the Point $D$. Make $PQ=a$, $DE=b$, $PT=c$, and $QE=x$, and $PD$ will be $=a+x-b$, as above. And the Time wherein the upper Globe, by falling, will deſcribe the Spaces $PT$ and $TD$, and the lower Globe by falling before, and then by re-aſcending, will deſcribe the Sum of the Spaces $QE+ED$ will be as $\sqrt{PT}$, $\sqrt{PD}-\sqrt{PT}$, and $2\sqrt{QE}-\sqrt{QD}$; that is, as the $\sqrt{c}$, $\sqrt{a+x-b}-\sqrt{c}$, and $2\sqrt{x}-\sqrt{x-b}$, but the two laſt Times, becauſe the Spaces $TD$ and $QE+ED$ are deſcrib'd together, are equal. Therefore $\sqrt{a+x-b}-\sqrt{c}=2\sqrt{x}-\sqrt{x-b}$. And the Parts being ſquar'd $a+c-2\sqrt{ca+cx-cb}=4x-4\sqrt{xx-bx}$. Make $a+c=e$, and $a-b=f$, and by a due Reduction $4x-e+2\sqrt{cf+cx}=4\sqrt{xx-bx}$, and the Parts being ſquar'd $ee-8ex+16xx+4cf+4cx+\overline{16x-4e}\times\sqrt{cf+cx}=16xx-16bx$. And blotting out on both Sides $16xx$, and writing $m$ for $ee+4ef$, and alſo writing $n$ for $8e-16b-4c$, you'll have by due Reduction $\overline{16x-4e}\times\sqrt{cf+cx}=nx-m$. And the Parts being ſquar'd [you'll have] $256cfxx+256cx^3-128cefx-128cexx+16ceef+16ceex=nnxx-2mnx+mm$. And having order'd the Æquation

$$\begin{array}{llll} 256cx^3 & & & \\ +256cf & -128cef & & \\ -128ce\ \ xx & +16cee\ \ \ x & +6ceef & =0. \\ -nn & +2mn & -mm & \end{array}$$

By the Conſtruction of which Æquation $x$ or $QE$ will be given, to which if you add the given Diſtances $PQ$ and $EF$, you'll have the Height $PF$, which was to be found.

## PROBLEM LX.

*If two quieſcent Globes, the upper one A and the under one B, are let fall at different Times; and the lower Globe begins to fall in the ſame Moment that the upper one, by falling, has deſcrib'd the Space PT; to find the Places* $\alpha$, $\beta$, *which thoſe falling Globes ſhall occupy when their Interval or Diſtance* $\pi\chi$ *is given.* [Vide Figure 84.]

SINCE the Diſtances $PT$, $PQ$, and $\pi\chi$ are given, call the firſt $a$, the ſecond $b$, the third $c$, and for $P\pi$, or the Space that the ſuperior Globe deſcribes by falling before it comes to the Place ſought $\alpha$, put $x$. Now the Times wherein the upper Globe deſcribes the Spaces $PT$, $P\pi$, $T\pi$, and the lower one the Space $Q\chi$, are as $\sqrt{PT}$, $\sqrt{P\pi}$, $\sqrt{P\pi} - \sqrt{PT}$, and $\sqrt{Q\chi}$. The latter two of which Times, becauſe the Globes by falling together deſcribe the Spaces $T\pi$ and $Q\chi$, are equal. Whence alſo the $\sqrt{P\pi} - \sqrt{PT}$ will be equal to the $\sqrt{Q\chi}$. $P\pi$ was $=x$, and $PT=a$, and by adding $\pi\chi$ or $c$, to $P\pi$, and ſubtracting $PQ$, or $b$, from the Sum you'll have $Q\chi = x+c-b$. Wherefore ſubſtituting theſe, you'll have $\sqrt{x} - \sqrt{a} = \sqrt{x+c-b}$. And ſquaring both Sides of the Æquation, there will ariſe $x+a-2\sqrt{ax} = x+c-b$. And blotting out on both Sides $x$, and ordering the Æquation, you'll have $a+b-c = 2\sqrt{ax}$. And having ſquar'd the Parts, the Square of $a+b-c$ will be $=4ax$, and that Square divided by $4a$ will be $=x$. or $4a$ will be to $a+b-c$ as $a+b-c$ is to $x$. But from $x$ found, or $P\pi$, there is given the Place ſought, *viz.* $\alpha$ of the ſuperior Globe ſought. And by the Diſtance of the Places, there is alſo given the Place of the lower one $\beta$.

And hence, if you were to find the Point where the upper Globe, by falling, will at length fall upon the lower one; by putting the Diſtance $\pi\chi = 0$, or by extirpating $c$, ſay, $4a$ is to $a+b$ as $a+b$ is to $x$, or $P\pi$, and the Point $\pi$ will be that ſought.

And reciprocally, if that Point $\pi$, or $\chi$, in which the upper Globe falls upon the under one, be given, and you are to find the Place $T$, which the lower Point $P$ of the upper falling Globe poſſeſs'd, or was then in, when the lower Globe began to fall; becauſe $4a$ is to $a+b$ as $a+b$ is to $x$; or multiplying the Means and Extreams together, $4ax = aa + 2ab + bb$, and by due ordering of the Æquation $aa = 4ax - 2ab - bb$; extract the Square Root, and you'll have $a = 2x - b - 2\sqrt{xx - bx}$. Take therefore $V\pi$, a mean Proportional between $P\pi$ and $Q\pi$, and towards $V$ take $VT = VQ$, and $T$ will be the Point you ſeek. For $V\pi$ will be $= \sqrt{P\pi \times Q\pi}$, that is, $= \sqrt{x \times \overline{x - b}}$, or $= \sqrt{xx - bx}$; the double whereof ſubtracted from $2x - b$, or from $2P\pi - PQ$, that is, from $PQ + 2Q\pi$, leaves $PQ - 2VQ$, or $PV - VQ$ that is, $PT$.

If, laſtly, the lower of the Globes, after the upper has fallen upon the lower, and the lower, by their Shock upon one another, is accelerated, and the ſuperior one retarded, the Places are requir'd where, in falling, they ſhall acquire a Diſtance equal to a given right Line. In the firſt Place, you muſt ſeek the Place where the upper one falls upon the lower one; then from the known Magnitudes of the Globes, as alſo from their Celerities where they fall on each other, you muſt find the Celerities they ſhall have immediately after Reflection, after the ſame Way as in *Probl.* 12. Afterwards you muſt find the higheſt Places to which theſe Globes, if they were carry'd upwards, would aſcend, and thence the Spaces will be known, which the Globes will deſcribe by falling in [any] given Times after Reflection, as alſo the Difference of the Spaces; and reciprocally from that Difference aſſum'd, you may go back Analytically to the Spaces deſcrib'd in falling.

As if the upper Globe falls upon the lower one at the Point $\pi$, [*Vide Figure* 85] and after Reflection, the Celerity of the upper one downwards be ſo great, as if it were upwards, it would cauſe that Globe to aſcend through the Space $\pi N$; and the Celerity of the lower one downwards was ſo great, as that, if it were upwards, it would cauſe the lower one to aſcend through the Space $\pi M$; then the Times wherein the upper Globe would reciprocally deſcend through the Spaces $N\pi$, $NC$, and the inferior one through the Spaces $M\pi$, $MH$, would be as $\sqrt{N\pi}$, $\sqrt{NC}$, $\sqrt{M\pi}$, $\sqrt{MH}$; and conſequently the Times wherein the upper Globe would run the Space $\pi G$, and the lower one $\pi H$, would be as $\sqrt{NG} - \sqrt{N\pi}$, to $\sqrt{MH} - \sqrt{M\pi}$. Make thoſe Times equal, and the $\sqrt{NG} - \sqrt{N\pi}$ will be $= \sqrt{MH} - \sqrt{M\pi}$. And, moreover, ſince there is given the Diſtance $GH$, put $\pi G + GH = \pi H$. And by the Reduction of theſe two Æquations the Problem will be ſolv'd. As if $M\pi$ is $= a$, $N\pi = b$, $GH = c$, $\pi G = x$, you'll have, according to the latter Æquation, $x + c = \pi H$. Add $M\pi$, you'll have $MH = a + c + x$. To $\pi G$ add $N\pi$, and you'll have $NG = b + x$. Which being found, according to the former Æquation, $\sqrt{b+x} - \sqrt{b}$ will be $= \sqrt{a+c+x} - \sqrt{a}$. Write $e$ for $a + c$, and $\sqrt{f}$ for $\sqrt{a} + \sqrt{b}$, and the Æquation will be $\sqrt{b+x} = \sqrt{c+x} + \sqrt{f}$. And the Parts being ſquar'd $b + x = e + x + f + 2\sqrt{ef+fx}$, or $b - e - f = 2\sqrt{ef+fx}$. For $b - e - f$ write $g$, and you'll have $g = 2\sqrt{ef+fx}$, and the Parts being ſquar'd, $gg = 4ef + 4fx$, and by Reduction $\frac{gg}{4f} - e = x$.

PRO-

## PROBLEM LXI.

*If there are two Globes, A, B, whereof the upper one A falling from the Height G, ſtrikes upon another lower one B rebounding from the Ground H upwards; and theſe Globes ſo part from one another by Reflection, that the Globe A returns by Force of that Reflection to its former Altitude G, and that in the ſame Time that the lower Globe B returns to the Ground H; then the Globe A falls again, and ſtrikes again upon the Globe B, rebounding again back from the Ground; and after this rate the Globes always rebound from one another and return to the ſame Place: From the given Magnitude of the Globes, the Poſition of the Ground, and the Place G from whence the upper Globe falls, to find the Place where the Globes ſhall ſtrike upon each other.* [Vide Figure 86.]

LET $e$ be the Center of the Globe *A*, and $f$ the Center of the Globe *B*, $d$ the Center of the Place *G* wherein the upper Globe is in its greateſt Height, $g$ the Center of the Place of the lower Globe where it falls on the Ground, $a$ the Semi-Diameter of the Globe *A*, $b$ the Semi-Diameter of the Globe *B*, $c$ the Point of Contact of the Globes falling upon one another, and *H* the Point of Contact of the lower Globe and the Ground. And the Swiftneſs of the Globe *A*, where it falls on the Globe *B*, will be the ſame which is generated by the Fall of the Globe from the Height $de$, and conſequently is as $\sqrt{de}$. With this ſame Celerity the Globe *A* ought to be reflected upwards, that it may return to its former Place *G*. And the Globe *B* ought to be reflected with the ſame Celerity downwards wherewith it aſcended, that it may return in the ſame Time to the Ground it had mounted up from. And that both theſe may come to paſs, the Motion of the Globes in reflecting ought to be equal. But the Motions are compounded of the Celerities and Magnitudes together, and conſequently the Product of the Bulk and Celerity of one Globe will be equal to the Product of the Bulk and Celerity of the other. Whence, if the Product of the Bulk and Celerity of one Globe be divided by the Bulk of the other Globe, you'll have the Celerity of the other before and after Reflection, or at the End of the Aſcent, and at the Beginning of the Deſcent. Therefore this Celerity will be as $\frac{A\sqrt{de}}{B}$, or ſince the Globes are as the Cubes of the Radii as $\frac{a^3\sqrt{de}}{b^3}$. But as the Square of this Celerity to the Square of the Celerity of the Globe $A$ juſt before Reflection, ſo would be the Height to which the Globe $B$ would aſcend with this Celerity, if it was not hinder'd by meeting the Globe $A$ falling upon it, to the Height $ed$ from which the Globe $B$ deſcends. That is, as $\frac{Aq}{Bq}de$ to $de$, or as $Aq$ to $Bq$, or $a^6$ to $b^6$, ſo that firſt Height to $x$, if you put $x$ for the latter Height $ed$. Therefore this Height, *viz.* to which $B$ would aſcend, if it was not hinder'd, is $\frac{a^6}{b^6}x$. Let that be $fK$. To $fK$ add $fg$, or $dH - ef - gH$; that is, $p - x$, if for the given $dH - ef - gH$ you write $p$, and $x$ for the unknown $de$; and you'll have $Kg = \frac{a^6}{b^6}x + p - x$. Whence the Celerity of the Globe $B$, when it falls from $K$ to the Ground, that is when it falls through the Space $Kg$, which its Centre would deſcribe in falling, will be as $\sqrt{\frac{a^6}{b^6}x + p - x}$. But that Globe falls from the Place $Bcf$ to the Ground in the ſame Time that the upper Globe $A$ aſcends from the Place $Ace$ to its greateſt Height $d$, or on the other Hand falls from $d$ to the Place $Ace$; and then ſince the Celerities of falling Bodies are equally augmented in equal Times, the Celerity of the Globe $B$, by falling to the Ground, will be augmented as much as is the whole Celerity which the Globe $A$ acquires by falling in the ſame Time from $d$ to $e$, or loſes by aſcending from $e$ to $d$. Therefore, to the Celerity which the Globe $B$ has in the Place $Bcf$, add the Celerity which the Globe

*A* has in the Place *Ace*, and the Sum, which is as $\sqrt{de}+\frac{a^3\sqrt{de}}{b^3}$, or $\sqrt{x}+\frac{a^3}{b^3}\sqrt{x}$, will be the Celerity of the Globe *B* where it falls on the Ground. Then the $\sqrt{x}+\frac{a^3}{b^3}\sqrt{x}$ will be equal to $\sqrt{\frac{a^6}{b^6}x+p-x}$. For $\frac{a^3+b^3}{b^3}$ write $\frac{r}{s}$, and for $\frac{a^6-b^6}{b^6}$, $\frac{rt}{ss}$, and that Æquation will become $\frac{r}{s}\sqrt{x}=\sqrt{\frac{rt}{ss}x+p}$, and the Parts being ſquar'd, $\frac{rr}{ss}x=\frac{rt}{ss}x+p$, ſubtract from both Sides $\frac{rt}{ss}x$, and multiply all into $ss$, and divide by $rr-rt$, and there will ariſe $x=\frac{ssp}{rr-rt}$. Which Æquation would have come out more ſimple, if I had taken $\frac{p}{s}$ for $\frac{a^3+b^3}{b^3}$, for there would have come out $\frac{ss}{p-t}=x$. Whence making that $p-t$ ſhall be to $s$ as $s$ to $x$, you'll have $x$, or $ed$; to which if you add $ec$, you'll have $dc$, and the Point $c$, in which the Globes ſhall fall upon one another. Q. E. F.

Hitherto I have been ſolving ſeveral Problems. For in learning the Sciences, Examples are of more Uſe than Precepts. Wherefore I have been the larger on this Head. And ſome which occurr'd as I was putting down the reſt, I have given their Solutions without uſing Algebra, that I might ſhew that in ſome Problems that at firſt Sight appear difficult, there is not always Occaſion for Algebra. But now it is Time to ſhew the Solution of Æquations. For after a Problem is brought to an Æquation, you muſt extract the Roots of that Æquation, which are the Quantities that [anſwer or] ſatisfy the Problem.

## *How* ÆQUATIONS *are to be ſolv'd.*

AFTER therefore in the Solution of a Queſtion you are come to an Æquation, and that Æquation is duly reduc'd and order'd; when the Quantities which are ſuppos'd given, are really given in Numbers, thoſe Numbers are to be ſubſtituted in their room in the Æquation, and you'll have a Numeral Æquation, whoſe Root being extracted will ſatisfy the Queſtion. As if in the Diviſion of an Angle into five equal Parts, by putting $r$ for the Radius of the Circle, $q$ for the Chord of the Complement of the propos'd Angle to two right ones, and $x$ for the Chord of the Complement of the fifth Part of that Angle, I had come to this Æquation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Caſe the Radius $r$ is given in Numbers, and the Line $q$ ſubtending the Complement of the given Angle; as if Radius were 10, and the Chord 3; I ſubſtitute thoſe Numbers in the Æquation for $r$ and $q$, and there comes out the Numeral Æquation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be $x$, or the Line ſubtending the Complement of the fifth Part of that given Angle.

*Of the Nature of the Roots of an Æquation.*

But the Root is a Number which being ſubſtituted in the Æquation for the Letter or Species ſignifying the Root, will make all the Terms vaniſh. Thus Unity is the Root of the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, becauſe being writ for $x$ it produces $1 - 1 - 19 + 49 - 30$, that is, nothing. And thus, if for $x$ you write the Number 3, or the Negative Number $-5$, and in both Caſes there will be produc'd nothing, the Affirmative and Negative Terms in theſe four Caſes deſtroying one another; then ſince any of the Numbers written in the Æquation fulfils the Condition of $x$, by making all the Terms of the Æquation together equal to nothing, any of them will be the Root of the Æquation.

And that you may not wonder that the ſame Æquation may have ſeveral Roots, you muſt know that there may be more Solutions [than one] of the ſame Problem. As if there was ſought the Interſection of two given Circles; there are two Interſections, and conſequently the Queſtion admits two Anſwers; and then the Æquation determining the Interſection will have two Roots, whereby it determines both [Points of] the Interſection, if there be nothing in the Data whereby the Anſwer is determin'd to [only] one Interſection. [*Vide Figure* 87.] And thus, if the Arch *APB* the fifth Part of *AP* were to be found, though perhaps you might apply your Thoughts only to the Arch *APB*, yet the Æquation, whereby the Queſtion will be ſolv'd, will determine the fifth Part of all the Arches which are terminated at the Points *A* and *B*; *viz.* the fifth Part of the Arches *ASB*, *APBSAPB*, *ASBPASB*, and *APBSAPBSAPB*, as well as the fifth Part of the Arch *APB*; which fifth Part, if you divide the whole Circumference into five equal Parts *PQ*, *QR*, *RS*, *ST*, *TP*, will be *AT*, *AQ*, *ATS*, *AQR*. Wherefore, by ſeeking the fifth Parts of the Arches which the right Line *AB* ſubtends, to determine all the Caſes the whole Circumference ought to be divided in the five Points *P*, *Q*, *R*, *S*, *T*. Wherefore, the Æquation that will determine all the Caſes will have five Roots. For the fifth Parts of all theſe Arches depend on the ſame Data, and are found by the ſame Kind of Calculus; ſo that you'll always fall upon the ſame Æquation, whether you ſeek the fifth Part of the Arch *APB*, or the fifth Part of the Arch *ASB*, or the fifth Part of any other of the Arches. Whence, if the Æquation by which the fifth Part of the Arch *APB* is determin'd, ſhould not have more than one Root, while by ſeeking the fifth Part of the Arch *ASB* we fall upon that ſame Æquation; it would follow, that this greater Arch would have the ſame fifth Part with the former, which is leſs, becauſe its Subtenſe [or Chord] is expreſs'd by the ſame Root of the Æquation. In every Problem therefore it is neceſſary, that the Æquation which anſwers ſhould have as many Roots as there are different Caſes of the Quantity ſought depending on the ſame Data, and to be determin'd by the ſame Method of Reaſoning.

But an Æquation may have as many Roots as it has Dimenſions, and not more. Thus the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, has four Roots, 1, 2, 3, −5, but not more. For any of theſe Numbers writ in the Æquation for $x$ will cauſe all the Terms to deſtroy one another as we have ſaid; but beſides theſe, there is no Number by whoſe Subſtitution this will happen. Moreover, the Number and Nature of the Roots will be beſt underſtood from the Generation of the Æquation. As if we would know how an Æquation is generated, whoſe Roots are 1, 2, 3, and — 5; we are to ſuppoſe $x$ to ſignify ambiguouſly thoſe Numbers, or $x$ to be $=1$, $x=2$, $x=3$, and $x=-5$, or which is the ſame Thing, $x-1=0$, $x-2=0$, $x-3=0$, and $x+5=0$; and multiplying theſe together, there will come out by the Multiplication of $x-1$ by $x-2$ this Æquation $xx-3x+2=0$, which is of two Dimenſions, and has two Roots 1 and 2. And by the Multiplication of this by $x-3$, there will come out $x^3-6xx+11x-6=0$, an Æquation of three Dimenſions and as many Roots; which again multiply'd by $x+5$ becomes $x^4-x^3-19xx+49x-30=0$, as above. Since therefore this Æquation is generated by four Factors, $x-1$, $x-2$, $x-3$, $x+5$, continually multiply'd by one another, where any of the Factors is nothing, that which is made by all will be nothing; but where none of them is nothing, that which is contain'd under them all cannot be nothing. That is, $x^4-x^3-19xx+49x-30$ cannot be $=0$, as ought to be, except in theſe four Caſes, where $x-1$ is $=0$, or $x-2=0$, or $x-3=0$, or, laſtly, $x+5=0$, and then only the Numbers 1, 2, 3, and $-5$ can exhibit $x$, or be the Roots of the Æquation. And you are to reaſon alike of all Æquations. For we may imagine all to be generated by ſuch a Multiplication, although it is uſually very difficult to ſeparate the Factors from one another, and is the ſame Thing as to reſolve the Æquation and extract its Roots. For the Roots being had, the Factors are had alſo.

But the Roots are of two Sorts, Affirmative, as in the Example brought, 1, 2, and 3, and Negative, as $-5$. And of theſe ſome are often impoſſible. Thus, the two Roots of the Æquation $xx-2ax+bb=0$, which are $a+\sqrt{aa-bb}$ and $a-\sqrt{aa-bb}$, are real when $aa$ is greater than $bb$; but when $aa$ is leſs than $bb$, they become impoſſible, becauſe then $aa-bb$ will be a Negative Quantity, and the Square Root of a Negative Quantity is impoſſible. For every poſſible Root, whether it be Affirmative or Negative, if it be multiply'd by it ſelf, produces an Affirmative Square; therefore that will be an impoſſible one which is to produce a Negative Square. By the ſame Argument you may conclude, that the Æquation $x^3-4xx+4x-6=0$, has one real Root, which is 2, and two impoſſible ones $1+\sqrt{-2}$ and $1-\sqrt{-2}$. For any of theſe, 2, $1+\sqrt{-2}$, $1-\sqrt{-2}$ being writ in the Æquation for $x$, will make all its Terms deſtroy one another; but $1+\sqrt{-2}$, and $1-\sqrt{-2}$, are im-

impoſſible Numbers, becauſe they ſuppoſe the Extraction of the Square Root out of the Negative Number — 2.

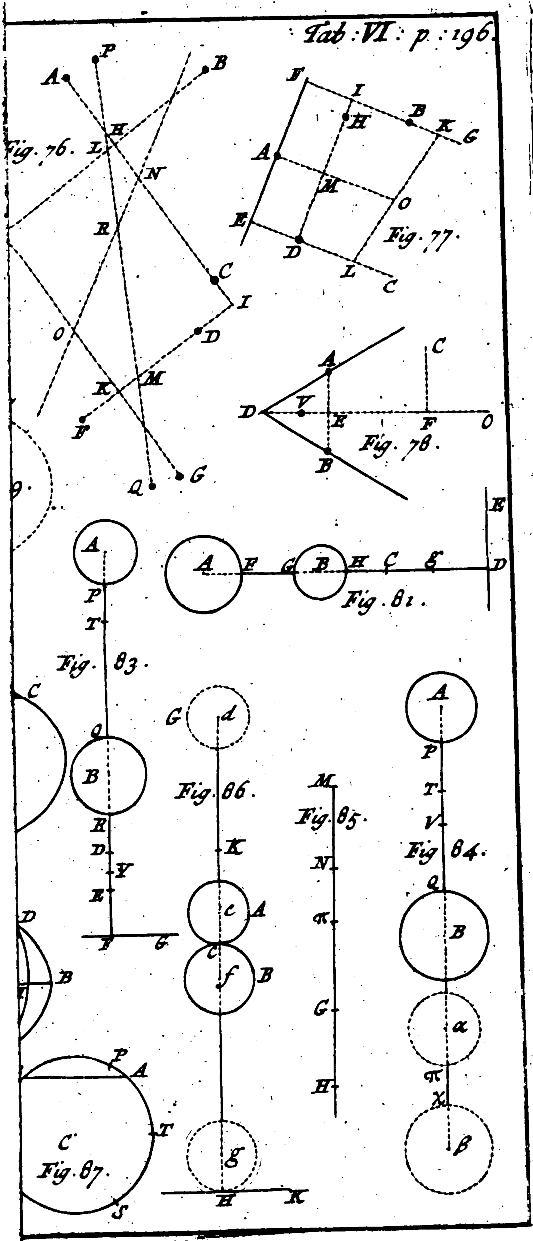
But it is juſt, that the Roots of Æquations ſhould be often impoſſible, leſt they ſhould exhibit the Caſes of Problems that are often impoſſible as if they were poſſible. As if you were to determine the Interſection of a right Line and a Circle, and you ſhould put two Letters for the Radius of the Circle and the Diſtance of the right Line from its Center; and when you have the Æquation defining the Interſection, if for the Letter denoting the Diſtance of the right Line from the Center, you put a Number leſs than the Radius, the Interſection will be poſſible; but if it be greater, impoſſible; and the two Roots of the Æquation, which determine the two Interſections, ought to be either poſſible or impoſſible, that they may truly expreſs the Matter. [*Vide Figure* 88.] And thus, if the Circle *CDEF*, and the Ellipſis *ACBF*, cut one another in the Points *C*, *D*, *E*, *F*, and to any right Line given in Poſition, as *AB*, you let fall the Perpendiculars *CG*, *DH*, *EI*, *FK*, and by ſeeking the Length of any one of the Perpendiculars, you come at length to an Æquation; that Æquation, where the Circle cuts the Ellipſis in four Points, will have four real Roots, which will be thoſe four Perpendiculars. Now, if the Radius of the Circle, its Center remaining, be diminiſh'd until the Points *E* and *F* meeting, the Circle at length touches the Ellipſe; thoſe two of the Roots which expreſs the Perpendiculars *EI* and *FK* now coinciding, will become equal. And if the Circle be yet diminiſh'd, ſo that it does not touch the Ellipſe in the Point *EF*, but only cuts it in the other two Points *C*, *D*, then out of the four Roots thoſe two which expreſs'd the Perpendiculars *EI*, *FK*, which are now become impoſſible, will become, together with thoſe Perpendiculars, alſo impoſſible. And after this Way in all Æquations, by augmenting or diminiſhing their Terms of the unequal Roots, two will become firſt equal and then impoſſible. And thence it is, that the Number of the impoſſible Roots is always even.

But ſometimes the Roots of Æquations are poſſible, when the Schemes exhibit them as impoſſible. But this happens by reaſon of ſome Limitation in the Scheme, which does not belong to the Æquation. [*Vide Figure* 89.] As if in the Semi-Circle *ADB*, having given the Diameter *AB*, and the Chord *AD*, and having let fall the Perpendicular *DC*, I was to find the Segment of the Diameter *AC*, you'll have $\frac{ADq}{AB}=AC$. And, by this Æquation, $AC$ is exhibited a real Quantity, where the inſcrib'd Line $AD$ is greater than the Diameter $AB$; but by the Scheme, $AC$ then becomes impoſſible, *viz.* in the Scheme the Line $AD$ is ſuppos'd to be inſcrib'd in the Circle, and therefore cannot be greater than the Diameter of the Circle; but in the Æquation there is nothing that depends upon that Condition. From this Condition alone of the Lines the Æquation comes out, that $AB$, $AD$, and $AC$ are continually proportional. And becauſe the Æquation does not contain all the Conditions of the Scheme, it is not neceſſary that it ſhould be bound to the Limits of all the Conditions. Whatever is more in the Scheme than in the Æquation may conſtrain that to Limits, but not this. For which reaſon, when Æquations are of odd Dimenſions, and conſequently cannot have all their Roots impoſſible, the Schemes often ſet Limits to the Quantities on which all the Roots depend, which 'tis impoſſible they can exceed, keeping the ſame Conditions of the Schemes.

Of thoſe Roots that are real ones, the Affirmative and Negative ones lie on contrary Sides, or tend contrary Ways. Thus, in the laſt Scheme but one. by ſeeking the Perpendicular $CG$, you'll light upon an Æquation that has two Affirmative Roots $CG$ and $DH$, tending from the Points $C$ and $D$ the ſame Way; and two Negative ones, $EI$ and $FK$, tending from the Points $E$ and $F$ the oppoſite Way. Or if in the Line $AB$ there be given any Point $P$, and the Part of it $PG$ extending from that given Point to ſome of the Perpendiculars, as $CG$, be ſought, we ſhall light on an Æquation of four Roots, $PG$, $PH$, $PI$, and $PK$, whereof the Quantity ſought $PG$, and thoſe that tend from the Point $P$ the ſame Way with $PG$, (as $PK$) will be Affirmative, but thoſe which tend the contrary Way (as $PH$, $PI$) Negative.

Where there are none of the Roots of the Æquation impoſſible, the Number of the Affirmative and Negative Roots may be known from the Signs of the Terms of the Æquation. For there are ſo many Affirmative Roots as there are Changes of the Signs in a continual Series from + to —, and from — to +; the reſt are Negative. As in the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, where the Signs of the Terms follow one another in this Order, + — —

Tab: VI: p: 196.



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$+-$, the Variations of the ſecond $-$ from the firſt $+$, of the fourth $+$ from the third $-$, and of the fifth $-$ from the fourth $+$, ſhew, that there are three Affirmative Roots, and conſequently, that the fourth is a Negative one. But where ſome of the Roots are impoſſible, the Rule is of no Force, unleſs as far as thoſe impoſſible Roots, which are neither Negative nor Affirmative, may be taken for ambiguous ones. Thus in the Æquation $x^3+pxx+3ppx-q=0$, the Signs ſhew that there is one Affirmative Root and two Negative ones. Suppoſe $x=2p$, or $x-2p=0$, and multiply the former Æquation by this, $x-2p=0$, and add one Affirmative Root more to the former, and you'll have this Æquation, $x^4-px^3+ppxx\begin{matrix}-2p^3\\-q\end{matrix}x+2pq=0$, which ought to have two Affirmative and two Negative Roots; yet it has, if you regard the Change of the Signs, four Affirmative ones. There are therefore two impoſſible ones, which for their Ambiguity in the former Caſe ſeem to be Negative ones, in the latter, Affirmative ones.

But you may know almoſt by this Rule how many Roots are impoſſible. Make a Series of Fractions, whoſe Denominators are Numbers in this Progreſſion 1, 2, 3, 4, 5, &c. going on to the Number which ſhall be the ſame as that of the Dimenſions of the Æquation; and the Numerators the ſame Series of Numbers in a contrary Order. Divide each of the latter Fractions by each of the former, and place the Fractions that come out on the middle Terms of the Æquation. And under any of the middle Terms, if its Square, multiply'd into the Fraction ſtanding over its Head, is greater than the Rectangle of the Terms on both Sides, place the Sign $+$, but if it be leſs, the Sign $-$; but under the firſt and laſt Term place the Sign $+$. And there will be as many impoſſible Roots as there are Changes in the Series of the underwritten Signs from $+$ to $-$, and $-$ to $+$. As if you have the Æquation $x^3+pxx+3ppx-q=0$; I divide the ſecond of the Fractions of this Series $\frac{3}{1}.\frac{2}{2}.\frac{1}{3}$, *viz.* $\frac{2}{2}$ by the firſt $\frac{3}{1}$, and the third $\frac{1}{3}$ by the ſecond $\frac{2}{2}$, and I place the Fractions that come out, *viz.* $\frac{1}{3}$ and $\frac{1}{3}$ upon the mean Terms of the Æquation, as follows;

$$\begin{array}{ccccccc} & & \frac{1}{3} & & \frac{1}{3} & & \\ x^3 & + & pxx & + & 3ppx & - & q=0. \\ + & & - & & + & & + \end{array}$$

Then, becauſe the Square of the ſecond Term $pxx$ multiply'd into the Fraction over its Head $\frac{1}{3}$, *viz.* $\frac{ppx^4}{3}$ is leſs than $3ppx^4$, the Rectangle of the firſt Term $x^3$ and third $3ppx$, I place the Sign $-$ under the Term $pxx$. But becauſe $9p^4xx$ (the Square of the third Term $3ppx$) multiply'd into the Fraction over its Head $\frac{1}{3}$, is greater than nothing and therefore much greater than the Negative Rectangle of the ſecond Term $pxx$, and the fourth $-q$, I place the Sign $+$ under that third Term. Then, under the firſt Term $x^3$ and the laſt $-q$, I place the Sign $+$. And the two Changes of the underwritten Signs; which are in this Series $+-++$, the one from $+$ into $-$; and the other from $-$ into $+$, ſhew that there are two impoſſible Roots. And thus the Æquation $x^3-4xx+4x-6=0$ has two impoſſible Roots,

$$\underset{+}{x^3} - \underset{+}{\overset{\frac{1}{3}}{4xx}} + \underset{-}{\overset{\frac{1}{3}}{4x}} - \underset{+}{6} = 0.$$

Alſo the Æquation $x^4-6xx-3x-2=0$ has two,

$$\underset{+}{x^4}\ \underset{+}{\overset{\frac{3}{8}}{*}} - \underset{+}{\overset{\frac{4}{9}}{6xx}} - \underset{-}{\overset{\frac{3}{8}}{3x}} - \underset{+}{2} = 0.$$

For this Series of Fractions $\frac{1}{4}\cdot\frac{2}{3}\cdot\frac{3}{2}$, by dividing the ſecond by the firſt, and the third by the ſecond, and the fourth by the third, gives this Series $\frac{3}{8}\cdot\frac{4}{9}\cdot\frac{3}{8}$, to be placed upon the middle Terms of the Æquation. Then the Square of the ſecond Term, which is here nothing, multiply'd into the Fraction over Head, *viz.* $\frac{3}{8}$, produces nothing, which is yet greater than the Negative Rectangle $-6x^6$ contain'd under the Terms $x^4$ and $-6xx$. Wherefore, under the Term that is wanting I write $+$. In the reſt I go on as in the former Example; and there comes out this Series of the underwritten Signs $+++-+$, where two Changes ſhew there are two impoſſible Roots. And after the ſame Way in the Æquation $x^5-4x^4+4x^3-2xx-5x-4=0$, are diſcover'd two impoſſible Roots, as follows;

$$\underset{+}{x^5} - \underset{+}{\overset{\frac{2}{5}}{4x^4}} + \underset{-}{\overset{\frac{1}{2}}{4x^3}} - \underset{+}{\overset{\frac{1}{2}}{2xx}} - \underset{+}{\overset{\frac{2}{5}}{5x}} - \underset{+}{4} = 0.$$

Where two or more Terms are at once wanting, under the firſt of the deficient Terms you muſt write the Sign $-$, under the ſecond the Sign $+$, under the third the Sign $-$, and ſo on, always varying the Signs, except that under the laſt of the deficient Terms you muſt always place +, where the Terms next on both Sides the deficient Terms have contrary Signs. As in the Æquations $\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & a^5=0, \\ + & + & - & + & - & + \end{array}$ and $\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & -a^5=0 \\ + & + & - & + & + & + \end{array}$; the firſt whereof has four, and the latter two impoſſible Roots. Thus alſo the Æquation,

$$\begin{array}{cccccccc} & \frac{3}{7} & \frac{5}{9} & \frac{3}{5} & \frac{3}{5} & \frac{5}{9} & \frac{3}{7} & \\ x^7 & -2x^6 & +3x^5 & -2x^4 & +x^3 & * & * & -3=0 \\ + & - & + & - & + & - & + & + \end{array}$$

has ſix impoſſible Roots.

Hence alſo may be known whether the impoſſible Roots are among the Affirmative or Negative ones. For the Signs of the Terms over Head of the ſubſcrib'd changing Terms ſhew, that there as many impoſſible Affirmative [Roots] as there are Variations of them, and as many Negative ones as there are Succeſſions without Variations. Thus, in the Æquation $\begin{array}{cccccc} x^5 & -4x^4 & +4x^3 & -2xx & -5x & -4=0 \\ + & + & - & + & + & + \end{array}$, becauſe by the Signs that are writ underneath that are changeable, *viz.* + − +, by which it is ſhewn there are two impoſſible Roots, the Terms over Head $-4x^4+4x^3-2xx$ have the Signs − + −, which by two Variations ſhew there are two Affirmative Roots; therefore there will be two impoſſible Roots among the Affirmative ones. Since therefore the Signs of all the Terms of the Æquation + − + − − −, by three Variations ſhew that there are three Affirmative Roots, and that the other two are Negative, and that among the Affirmative ones there are two impoſſible ones; it follows that there are, *viz.* one true affirmative Root, two negative ones, and two impoſſible ones. Now, if the Æquation had been $\begin{array}{cccccc} x^5 & -4x^4 & -4x^3 & -2xx & -5x & -4=0 \\ + & + & - & - & + & + \end{array}$, then the Terms over Head of the ſubſcrib'd former Terms + −, *viz.* $-4x^4-4x^3$, by their Signs that don't change − and −, ſhew, that one of the Negative Roots is impoſſible; and the Terms over the former underwritten varying Terms − +, *viz.* $-2xx-5x$, by their Terms not varying, − and −, ſhew that one of the Negative Roots are impoſſible. Wherefore, ſince the Signs of the Æquation + − − − − − by one Variation ſhew there is one Affirmative Root, and that the other four are Negative;

Negative, it follows, there is one Affirmative, two Negative, and two Impoſſible ones. And this is ſo where there are not more impoſſible Roots than what are diſcover'd by the Rule preceding. For there may be more, although it ſeldom happens.

*Of the Tranſmutations of Æquations.*

Moreover, all the Affirmative Roots of any Æquation may be chang'd into Negative ones, and the Negative into Affirmative ones, and that only by changing of the alternate Terms [*i. e.* every other Term]. Thus, in the Æquation $x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 = 0$, the three Affirmative Roots will be chang'd into Negative ones, and the two Negative ones into Affirmatives, by changing only the Signs of the ſecond, fourth, and ſixth Terms, as is done here, $x^5 + 4x^4 + 4x^3 + 2xx - 5x + 4 = 0$. This Æquation has the ſame Roots with the former, unleſs that in this, thoſe Roots are Affirmative that were there Negative, and Negative here that there were Affirmative; and the two impoſſible Roots, which lay hid there among the Affirmative ones, lie hid here among the Negative ones; ſo that theſe being ſubduc'd, there remains only one Root truly Negative.

There are alſo other Tranſmutations of Æquations which are of Uſe in divers Caſes. For we may ſuppoſe the Root of an Æquation to be compos'd any how out of a known and an unknown Quantity, and then ſubſtitute what we ſuppoſe equivalent to it. As if we ſuppoſe the Root to be equal to the Sum or Difference of any known and unknown Quantity. For, after this Rate, we may augment or diminiſh the Roots of the Æquation by that known Quantity, or ſubtract them from it; and thereby cauſe that ſome of them that were before Negative ſhall now become Affirmative, or ſome of the Affirmative ones become Negative, or alſo that all ſhall become Affirmative or all Negative. Thus, in the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, if I have a mind to augment the Roots by Unity, I ſuppoſe $x + 1 = y$, or $x = y - 1$; and then for $x$ I write $y - 1$ in the Æquation, and for the Square, Cube, or Biquadrate of $x$, I write the like Power of $y - 1$, as follows;

$$\begin{array}{r|r}
x^4 & y^4 - 4y^3 + 6yy - 4y + 1 \\
-x^3 & -y^3 + 3yy - 3y + 1 \\
-19xx & -19yy + 38y - 19 \\
+49x & +49y - 49 \\
-30 & -30 \\
\hline
\text{Sum} & y^4 - 5y^3 - 10yy + 80y - 96 = 0.
\end{array}$$

And the Roots of the Æquation that is produc'd, (*viz.*) $y^4 - 5y^3 - 10yy + 80y - 96 = 0$, will be 2, 3, 4, $-4$, which before were 1, 2, 3, $-5$, *i. e.* bigger by Unity. Now, if for $x$ I had writ $y + 1\frac{1}{2}$, there would have come out the Æquation $y^4 + 5y^3 - 10yy - \frac{5}{4}y + \frac{39}{16} = 0$, whereof there be two Affirmative Roots $\frac{1}{2}$ and $1\frac{1}{2}$, and two Negative ones, $-\frac{1}{2}$ and $-6\frac{1}{2}$. But by writing $y - 6$ for $x$, there would have come out an Æquation whoſe Roots would have been 7, 8, 9, 1, *viz.* all Affirmative; and writing for the ſame [$x$] $y + 4$, there would have come out thoſe Roots diminiſh'd by 4, *viz.* $-3$ $-2$ $-1$ $-9$, all of them Negative.

And after this Way, by augmenting or diminiſhing the Roots, if any of them are impoſſible, they will ſometimes be more eaſily detected this Way than before.

Thus, in the Æquation $x^3 - 3aax - 3a^3 = 0$, there are no Roots that appear impoſſible by the preceding Rule; but if you augment the Roots by the Quantity $a$, writing $y - a$ for $x$, you may by that Rule diſcover two impoſſible Roots in the Æquation reſulting, $y^3 - 3ayy - a^3 = 0$.

By the ſame Operation you may alſo take away the ſecond Terms of Æquations; which will be done, if you ſubduct the known Quantity [or Co-efficient] of the ſecond Term of the Æquation propos'd, divided by the Number of Dimenſions [of the higheſt Term] of the Æquation, from the Quantity which you aſſume to ſignify the Root of the new Æquation, and ſubſtitute the Remainder for the Root of the Æquation propos'd. As if there was propos'd the Æquation $x^3 - 4xx + 4x - 6 = 0$, I ſubtract the known Quantity [or Co-efficient] of the ſecond Term, which is $-4$, divided by the Number of the Dimenſions of the Æquation, *viz.* 3, from the Species [or Letter] which is aſſum'd to ſignify the new Root, ſuppoſe from $y$, and the Remainder $y + \frac{4}{3}$ I ſubſtitute for $x$, and there comes out

$$\begin{array}{r}
y^3 + 4yy + \frac{16}{3}y + \frac{64}{27} \\
-4yy - \frac{32}{3}y - \frac{64}{9} \\
+4y + \frac{16}{3} \\
-6 \\
\hline
y^3 \quad * \quad -\frac{4}{3}y - \frac{146}{27} = 0.
\end{array}$$

By the ſame Method, the third Term of an Æquation may be alſo taken away. Let there be propos'd the Æquation $x^4 - 3x^3 + 3xx - 5x - 2 = 0$, and make $x = y - e$, and ſubſtituting $y - e$ in the room of $x$, there will ariſe this Æquation;

$$y^4 \begin{array}{l} -4e \\ -3 \end{array} y^3 \begin{array}{l} +6ee \\ +9e \\ +3 \end{array} yy \begin{array}{l} -4e^3 \\ -9ee \\ -6e \\ -5 \end{array} y \left.\begin{array}{l} +e^4 \\ +3e^3 \\ +3ee \\ +5e \\ -2 \end{array}\right\} = 0.$$

The third Term of this Æquation is $6ee + 9e + 3$ multiply'd by $yy$. Where, if $6ee + 9e + 3$ were nothing, you'd have what you deſir'd. Let us ſuppoſe it therefore to be nothing, that we may thence find what Number ought to be ſubſtituted in this Caſe for $e$, and we ſhall have the Quadratick Æquation $6ee + 9e + 3 = 0$, which divided by 6 will become $ee + \frac{3}{2}e + \frac{1}{2} = 0$, or $ee = -\frac{3}{2}e - \frac{1}{2}$, and extracting the Root $e = -\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}$, or $= -\frac{3}{4} \pm \sqrt{\frac{1}{16}}$, that is, $= -\frac{3}{4} \pm \frac{1}{4}$, and conſequently equal $\left\{\begin{array}{l} -\frac{1}{2} \\ -1 \end{array}\right\}$ Whence $y - e$ will be either $y + \frac{1}{2}$, or $y + 1$. Wherefore, ſince $y - e$ was writ for $x$; in the room of $y - e$ there ought to be writ $y + \frac{1}{2}$, or $y + 1$ for $x$, that the third Term of the Æquation that reſults may be taken away. And that will happen in both Caſes. For if for $x$ you write $y + \frac{1}{2}$, there will ariſe this Æquation, $y^4 - y^3 - \frac{15}{4}y - \frac{77}{16} = 0$; but if you write $y + 1$, there will ariſe this Æquation, $y^4 + y^3 - 4y - 12 = 0$.

Moreover, the Roots of Æquations may be multiply'd or divided by given Numbers; and after this Rate, the Terms of Æquations be diminiſh'd, and Fractions and Radical Quantities ſometimes be taken away. As if the Æquation were $y^3 - \frac{4}{3}y - \frac{146}{27} = 0$; in order to take away the Fractions, I ſuppoſe $y$ to be $= \frac{1}{3}z$, and then by ſubſtituting $\frac{1}{3}z$

for $y$, there comes out this new Æquation, $\frac{z^3}{27} - \frac{12z}{27} - \frac{146}{27} = 0$, and having rejected the common Denominator of the Terms, $z^3 - 12z - 146 = 0$, the Roots of which Æquation are thrice greater than before. And again, to diminiſh the Terms of this Æquation, if you write $2v$ for $z$, there will come out $8v^3 - 24v - 146 = 0$, and dividing all by 8, you'll have $v^3 - 3v - 18\frac{1}{4} = 0$; the Roots of which Æquation are half of the Roots of the former. And here, if at laſt you find $v$ make $2v = z$, $\frac{1}{3}z = y$, and $y + \frac{4}{3} = x$, and you'll have $x$ the Root of the Æquation as firſt propos'd.

And thus, in the Æquation $x^3 - 2x + \sqrt{3} = 0$, to take away the Radical Quantity $\sqrt{3}$; for $x$ I write $y\sqrt{3}$, and there comes out the Æquation $3y^3\sqrt{3} - 2y\sqrt{3} + \sqrt{3} = 0$, which, dividing all the Terms by the $\sqrt{3}$, becomes $3y^3 - 2y + 1 = 0$.

Again, the Roots of an Æquation may be chang'd into their Reciprocals, and after this Way the Æquation may be ſometimes reduc'd to a more commodious Form. Thus, the laſt Æquation $3y^3 - 2y + 1 = 0$, by writing $\frac{1}{z}$ for $y$, becomes $\frac{3}{z^3} - \frac{2}{z} + 1 = 0$, and all the Terms being multiply'd by $z^3$, and the Order of the Terms chang'd, $z^3 - 2zz + 3 = 0$. The laſt Term but one of an Æquation may alſo by this Method be taken away, as the ſecond was taken away before, as you ſee done in the precedent Æquation; or if you would take away the laſt but two, it may be done as you have taken away the third. Moreover, the leaſt Root may thus be converted into the greateſt, and the greateſt into the leaſt, which may be of ſome Uſe in what follows. Thus, in the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, whoſe Roots are 3, 2, 1, −5, if you write $\frac{1}{y}$ for $x$, there will come out the Æquation $\frac{1}{y^4} - \frac{1}{y^3} - \frac{19}{yy} + \frac{49}{y} - 30 = 0$, which, multiplying all the Terms by $y^4$, and dividing them by 30, the Signs being chang'd, becomes $y^4 - \frac{49}{30}y^3 + \frac{19}{30}y^2 + \frac{1}{30}y - \frac{1}{30} = 0$, the Roots whereof are $\frac{1}{3}$, $\frac{1}{2}$, 1, $-\frac{1}{5}$; the greateſt of the Affirmative Roots 3 being now changed into the leaſt $\frac{1}{3}$, and the leaſt 1 being now made greateſt, and the Negative Root $-5$, which of all was the moſt remote from 0, now coming neareſt to it.

There are alſo other Tranſmutations of Æquations, but which may all be perform'd after that Way of tranſmutating we have ſhewn, when we took away the third Term of the Æquation.

From the Generation of Æquations it is evident, that the known Quantity of the ſecond Term of the Æquation, if its Sign be chang'd, is equal to the Aggregate [or Sum] of all the Roots [added together] under their proper Signs; and that of the third Term equal to the Aggregate of the Rectangles of each two of the Roots; that of the fourth, if its Sign be chang'd, is equal to the Aggregate of the Contents under each three of the Roots; that of the fifth is equal to the Aggregate of the Contents under each four, and ſo on *ad infinitum*. Let us aſſume $x = a$, $x = b$, $x = -c$, $x = d$, &c. or $x - a = 0$, $x - b = 0$, $x + c = 0$, $x - d = 0$, and by the continual Multiplication of theſe we may generate Æquations as above. Now, by multiplying $x - a$ by $x - b$ there will be produc'd the Æquation $xx \begin{matrix}-a\\-b\end{matrix} x + ab = 0$; where the known Quantity of the ſecond Term, if its Signs are chang'd, *viz.* $a + b$, is the Sum of the two Roots $a$ and $b$, and the known Quantity of the third Term is the only Rectangle contain'd under both. Again, by multiplying this Æquation by $x + c$, there will be produc'd the Cubick Æquation $x^3 \begin{matrix}-a\\-b\\+c\end{matrix} xx \begin{matrix}+ab\\-ac\\-bc\end{matrix} x + abc = 0$, where the known Quantity of the ſecond Term having its Signs chang'd, *viz.* $a + b - c$, is the Sum of the Roots $a$, and $b$, and $-c$; the known Quantity of the third Term $ab - ac - bc$ is the Sum of the Rectangles under each two of the Terms $a$ and $b$, $a$ and $-c$, $b$ and $-c$; and the known Quantity of the fourth Term under its Sign chang'd, $-abc$, is the only Content generated by the continual Multiplication of all the Terms, $a$ by $b$ into $-c$. Moreover, by multiplying that Cubick Æquation by $x - d$, there will be produc'd this Biquadratick one;

$x^4$

$$x^4 \begin{array}{l} -a \\ -b \\ +c \\ -d \end{array} x^3 \begin{array}{l} +ab \\ -ac \\ -bc \\ +ad \\ +bd \\ -cd \end{array} xx \begin{array}{l} +abc \\ -abd \\ +bcd \\ +acd \end{array} x - abcd = 0.$$

Where the known Quantity of the ſecond Term under its Signs chang'd, *viz.* $a + b - c + d$, is the Sum of all the Roots; that of the third, $ab - ac - bc + ad + bd - cd$, is the Sum of the Rectangles under every two; that of the fourth, its Signs being chang'd, $-abc + abd - bcd - acd$, is the Sum of the Contents under each Ternary; that of the fifth, $-abcd$, is the only Content under them all. And hence we firſt infer, that all the rational Roots of any Æquation that involves neither Surds nor Fractions, and the Rectangles of any two of the Roots, or the Contents of any three or more of them, are ſome of the Integral Diviſors of the laſt Term; and therefore when it is evident, that there is no Diviſor of the laſt Term, or Root of the Æquation, or Rectangle, or Content of two or more, it will alſo be evident that there is no Root, or Rectangle, or Content of Roots, except what is Surd.

Let us ſuppoſe now, that the known Quantities of the Terms of [any] Æquation under their Signs chang'd, are $p$, $q$, $r$, $s$, $t$, $v$, &c. *viz.* that of the ſecond $p$, that of the third $q$, of the fourth $r$, of the fifth $s$, and ſo on. And the Signs of the Terms being rightly obſerv'd, make $p = a$, $pa + 2q = b$, $pb + qa + 3r = c$, $pc + qb + ra + 4s = d$, $pd + qc + rb + sa + 5t = e$, $pe + qd + rc + sb + ta + 6v = f$. and ſo on *ad infinitum*, obſerving the Series of the Progreſſion. And $a$ will be the Sum of the Roots, $b$ the Sum of the Squares of each of the Roots, $c$ the Sum of the Cubes, $d$ the Sum of the Biquadrates, $e$ the Sum of the Quadrato-Cubes, $f$ the Sum of the Cubo-Cubes [or ſixth Power] and ſo on. As in the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, where the known Quantity of the ſecond Term is $-1$, of the third $-19$, of the fourth $+49$, of the fifth $-30$; you muſt make $1 = p$, $19 = q$, $-49 = r$, $30 = s$. And there will thence ariſe $a = (p =)\ 1$, $b = (pa + 2q = 1 + 38 =)\ 39$, $c = (pb + qa + 3r = 39\ 19 - 147 =) -89$, $d = (pc + qb + ra + 4s = -89 + + 741 - 49 + 120 =)\ 723$. Wherefore the Sum of the

Roots will be 1, the Sum of the Squares of the Roots 39, the Sum of the Cubes $-89$, and the Sum of the Biquadrates 723, *viz.* the Roots of that Æquation are 1, 2, 3, and $-5$, and the Sum of theſe $1+2+3-5$ is 1; the Sum of the Squares, $1+4+9+25$, is 39; the Sum of the Cubes, $1+8+27-125$, is $-89$; and the Sum of the Biquadrates, $1+16+81+625$, is 723.

*Of the Limits of Æquations.*

And hence are collected the Limits between which the Roots of the Æquation ſhall conſiſt, if none of them is impoſſible. For when the Squares of all the Roots are Affirmative, the Sum of the Squares will be Affirmative, and therefore greater than the Square of the greateſt Root. And by the ſame Argument, the Sum of the Biquadrates of all the Roots will be greater than the Biquadrate of the greateſt Root, and the Sum of the Cubo-Cubes greater than the Cubo-Cube of the greateſt Root. Wherefore, if you deſire the Limit which no Roots can paſs, ſeek the Sum of the Squares of the Roots, and extract its Square Root. For this Root will be greater than the greateſt Root of the Æquation. But you'll come nearer the greateſt Root if you ſeek the Sum of the Biquadrates, and extract its Biquadratick Root; and yet nearer, if you ſeek the Sum of the Cubo-Cubes, and extract its Cubo-Cubical Root, and ſo on *ad infinitum.*

Thus, in the precedent Æquation, the Square Root of the Sum of the Squares of the Roots, or $\sqrt{39}$, is $6\frac{1}{2}$ nearly, and $6\frac{1}{2}$ is farther diſtant from 0 than any of the Roots 1, 2, 3, $-5$. But the Biquadratick Root of the Sum of the Biquadrates of the Roots, *viz.* $\sqrt[4]{723}$, which is $5\frac{1}{4}$ nearly, comes nearer to the Root that is moſt remote from nothing, *viz.* $-5$.

If, between the Sum of the Squares and the Sum of the Biquadrates of the Roots you find a mean Proportional, that will be a little greater than the Sum of the Cubes of the Roots connected under Affirmative Signs. And hence, *the* half Sum of this mean Proportional, and of the Sum of the Cubes collected under their proper Signs, found as before, will be greater than the Sum of the Cubes of the Affirmative Roots, and the half Difference greater than the Sum of the Cubes of the Negative Roots. And conſequently, the greateſt of the Affirmative Roots will be leſs than the Cube Root of that Semi-Difference. Thus, in the precedent Æquation, a mean Proportional between the Sum of the Squares of the Roots 39, and the Sum of the Biquadrates 723, is nearly 168. The Sum of the Cubes under their proper Signs was, as above, —89, the half Sum of this and 168 is $39\frac{1}{2}$, the Semi-Difference $128\frac{1}{2}$. The Cube Root of the former, which is about $3\frac{1}{2}$, is greater than the greateſt of the Affirmative Roots 3. The Cube Root of the latter, which is $5\frac{1}{21}$ nearly, is greater than the Negative Root —5. By which Example it may be ſeen how near you may come this Way to the Root, where there is only one Negative Root or one Affirmative one. And yet you might come nearer yet, if you found a mean Proportional between the Sum of the Biquadrates of the Roots and the Sum of the Cubo-Cubes, and if from the Semi-Sum and Semi-Difference of this, and of the Sum of the Quadrato-Cube of the Roots, you extracted the Quadrato-Cubical Roots. For the Quadrato-Cubical Root of the Semi-Sum would be greater than the greateſt Affirmative Root, and the Quadrato-Cubick Root of the Semi-Difference would be greater than the greateſt Negative Root, but by a leſs Exceſs than before. Since therefore any Root, by augmenting and diminiſhing all the Roots, may be made the leaſt, and then the leaſt converted into the greateſt, and afterwards all beſides the greateſt be made Negative, it is manifeſt how [any] Root deſired may be found nearly.

If all the Roots except two are Negative, thoſe two may be both together found this Way. The Sum of the Cubes of thoſe two Roots being found according to the precedent Method, as alſo the Sum of the Quadrato-Cubes, and the Sum of the Quadrato-Quadrato-Cubes of all the Roots: between the two latter Sums ſeek a mean Proportional, and that will be the Difference between the Sum of the Cubo-Cubes of the Affirmative Roots, and the Sum of the Cubo-Cubes of the Negative Roots nearly; and conſequently, the half Sum of this mean Proportional, and of the Sum of the Cubo-Cubes of all the Roots, will be the Semi-Sum of the Cubo-Cubes of the Affirmative Roots, and the Semi-Difference will be the Sum of the Cubo-Cubes of the Negative Roots. Having therefore both the Sum of the Cubes, and alſo the Sum of the Cubo-Cubes of the two Affirmative Roots, from the double of the latter Sum ſubtract the Square of the former Sum, and the Square Root of the Remainder will be the Difference of the Cubes of the two Roots. And having both the Sum and Difference of the Cubes, you'll have the Cubes themſelves. Extract their Cube Roots, and you'll nearly have the two Affirmative Roots of the Æquation. And if in higher Powers you ſhould do the like, you'll have the Roots yet more nearly. But theſe Limitations, by reaſon of the Difficulty of the Calculus, are of leſs Uſe, and extend only to thoſe Æquations that have no imaginary Roots, wherefore I will now ſhew how to find the Limits another Way, which is more eaſy, and extends to all Æquations.

Multiply every Term of the Æquation by the Number of its Dimenſions, and divide the Product by the Root of the Æquation; then again multiply every one of the Terms that come out by a Number leſs by Unity than before, and divide the Product by the Root of the Æquation, and ſo go on, by always multiplying by Numbers leſs by Unity than before, and dividing the Product by the Root, *till* at length all the Terms are deſtroy'd, whoſe Signs are different from the Sign of the firſt or higheſt Term, except the laſt; *and that* Number will be greater than any Affirmative Root; which being writ in the Terms that come out for [or in room of] *the* Root, makes the Aggregate of thoſe which were each Time produc'd by Multiplication to have always the ſame Sign with the firſt or higheſt Term of the Æquation. As if there was propos'd the Æquation $x^5\ 2x^4 - 10x^3 + 30xx + 63x - 120 = 0$. I firſt multiply this thus;

$$\overset{5}{x^5} - \overset{4}{2x^4} - \overset{3}{10x^3} + \overset{2}{30xx} + \overset{1}{63x} - \overset{0}{120}.$$

Then I again multiply the Terms that come out divided by $x$, thus;

$$\overset{4}{5x^4} - \overset{3}{8x^3} - \overset{2}{30xx} + \overset{1}{60x} + \overset{0}{63}.$$

And dividing the Terms that come out again by $x$, there comes out $20x^3 - 24xx - 60x + 60$; which, to leſſen them, I divide by the greateſt common Diviſor 4, and you have $5x^3 - 6xx - 15x + 15$. Theſe being again multiply'd by the Progreſſion 3, 2, 1, 0, and divided by $x$, becomes $5xx - 4x - 5$. And theſe multiply'd by the Progreſſion 2, 1, 0, and divided by $2x$ become $5x - 2$. Now, ſince the higheſt Term of the Æquation $x^5$ is Affirmative, I try what Number writ in theſe Products for $x$ will cauſe them all to be Affirmative. And by trying 1, you have $5x - 2 = 3$ Affirmative; but $5xx - 4x - 5$, you have $-4$ Negative. Wherefore the Limit will be greater than 1. I therefore try ſome greater Number, as 2; and ſubſtituting 2 in each for $x$, they become

$$\begin{array}{l} 5x-2=8 \\ 5xx-4x-5=7 \\ 5x^3-6xx-15x+15=1 \\ 5x^4-8x^3-30xx+60x+63=79 \\ x^5-2x^4-10x^3+30xx+63x-120=46. \end{array}$$

Wherefore, ſince the Numbers that come out 8 . 7 . 1 . 79 . 46 . are all Affirmative, the Number 2 will be greater than the greateſt of the Affirmative Roots. In like manner, if I would find the Limit of the Negative Roots, I try Negative Numbers. Or that which is all one, I change the Signs of every other Term, and try Affirmative ones. But having chang'd the Signs of every other Term, the Quantities in which the Numbers are to be ſubſtituted, will become

$$\begin{array}{l} 5x+2 \\ 5xx+4x-5 \\ 5x^3+6xx-15x-15 \\ 5x^4+8x^3-30xx-60x+63 \\ x^5+2x^4-10x^3-30xx+63x+120. \end{array}$$

Out of theſe I chuſe ſome Quantity wherein the Negative Terms ſeem moſt prevalent; ſuppoſe $5x^4+8x^3-30xx-60x+63$, and here ſubſtituting for $x$ the Numbers 1 and 2, there come out the Negative Numbers $-14$ and $-33$. Whence the Limit will be greater than $-2$. But ſubſtituting the Number 3, there comes out the Affirmative Number 234. And in like manner in the other Quantities, by ſubſtituting the Number 3 there comes out always an Affirmative Number, which may be ſeen by bare Inſpection. Wherefore the Number $-3$ is greater than all the Negative Roots. And ſo you have the Limits 2 and $-3$, between which are all the Roots.

But the Invention of Limits is of Uſe both in the Reduction of Æquations by Rational Roots, and in the Extraction of Surd Roots out of them; leaſt we might ſometimes go about to look for the Root beyond theſe Limits. Thus, in the laſt Æquation, if I would find the Rational Roots, if perhaps it has any; from what we have ſaid, it is certain they can be no other than the Diviſors of the laſt Term of the Æquation, which here is 120. Then trying all its Diviſors, if none of them writ in the Æquation for $x$ would make all the Terms vaniſh, it is certain that the Æquation will admit of no Root but what is Surd. But there are many Diviſors of the laſt Term 120, *viz.* 1. — 1. 2. — 2. 3. — 3. 4. — 4. 5. — 5. 6. — 6. 8. — 8. 10. — 10. 12. — 12. 15. — 15. 20. — 20. 24. — 24. 30. — 30. 40. — 40. 60. — 60. 120. and — 120. To try all theſe Diviſors would be tedious. But it being known that the Roots are between 2 and — 3, we are free'd from that Labour. For now there will be no need to try the Diviſors, unleſs thoſe only that are within theſe Limits, *viz.* the Diviſors 1, and — 1. and — 2. For if none of theſe are the Root, it is certain that the Æquation has no Root but what is Surd.

*The Reduction of Æquations by Surd Diviſors.*

Hitherto I have treated of the Reduction of Æquations which admit of Rational Diviſors; but before we can conclude, that an Æquation of four, ſix, or more Dimenſions is irreducible, we muſt firſt try whether or not it may be reduc'd by any Surd Diviſor; or, which is the ſame Thing, you muſt try whether the Æquation can be ſo divided into two equal Parts, that you can extract the Root out of both. But that may be done by the following Method.

Diſpoſe the Æquation according to the Dimenſion of ſome certain Letter, ſo that all its Terms jointly under their proper Signs, may be equal to nothing, and let the higheſt Term be affected with an Affirmative Sign. Then, if the Æquation be a Quadratick, (for we may add this Caſe for the Analogy of the Matter) take from both Sides the loweſt Term, and add one fourth Part of the Square of the known Quantity of the middle Term. As if the Æquation be $xx - ax - b = 0$, ſubtract from both Sides $-b$, and add $\frac{1}{4}aa$, and there will come out $xx - ax + \frac{1}{4}aa = b + \frac{1}{4}aa$, and extracting on both Sides the Root, you'll have $x - \frac{1}{2}a = +\sqrt{b + \frac{1}{4}aa}$, or $x = \frac{1}{2}a \pm \sqrt{b + \frac{1}{4}aa}$.

Now, if the Æquation be of four Dimenſions, ſuppoſe $x^4 + px^3 + qxx + rx + s = 0$, where $p$, $q$, $r$, and $s$ denote the known Quantities of the Terms of the Æquation affected by their proper Signs, make

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}\alpha = \beta.$$
$$s - \tfrac{1}{4}\alpha\alpha = \zeta.$$

Then put for $n$ ſome common Integral Diviſor of the Terms $\beta$ and $2\zeta$, that is not a Square, and which ought to be odd, and divided by 4 to leave Unity, if either of the Terms $p$ and $r$ be odd. Put alſo for $k$ ſome Diviſor of the

Quantity $\frac{\beta}{n}$ if $p$ be even; or half of the odd Diviſor, if $p$ be odd; or nothing, if the Dividual $\beta$ be nothing. Take the Quotient from $\frac{1}{2}pk$, and call the half of the Remainder $l$. Then for $Q$ put $\frac{\alpha+nkk}{2}$, and try if $n$ divides $QQ-s$, and the Root of the Quotient be rational and equal to $l$; which if it happen, add to each Part of the Æquation $nkkxx+2nklx+nll$, and extract the Root on both Sides, there coming out $xx+\frac{1}{2}px+Q=\sqrt{n}$ into $\overline{kx+l}$.

For Example, let there be propos'd the Æquation $x^4+12x-17=0$, and becauſe $p$ and $q$ are both here wanting, and $r$ is 12, and $s$ is $-17$, having ſubſtituted theſe Numbers, you'll have $\alpha=0$, $\beta=12$, and $\zeta=-17$, and the only common Diviſor of $\beta$ and $2\zeta$, viz. 2, will be $n$. Moreover, $\frac{\beta}{n}$ is 6, and its Diviſors 1, 2, 3, and 6, are ſucceſſively to be try'd for $k$, and $-3$, $-2$, $-1$, $-\frac{1}{2}$, for $l$ reſpectively. But $\frac{\alpha+nkk}{2}$, that is, $kk$ is equal to $Q$. Moreover, $\sqrt{\frac{QQ-s}{n}}$, that is, $\sqrt{\frac{QQ+17}{2}}$ is equal to $l$. Where the even Numbers 2 and 6 are writ for $k$, $Q$ is 4 and 36, and $QQ-s$ will be an odd Number, and conſequently cannot be divided by $n$ or 2. Wherefore thoſe Numbers 2 and 6 are to be rejected. But when 1 and 3 are writ for $k$, $Q$ is 1 and 9, and $QQ-s$ is 18 and 98, which Numbers may be divided by $n$, and the Roots of the Quotients extracted. For they are $\pm 3$ and $\pm 7$; whereof only $-3$ agrees with $l$. I put therefore $k=1$, $l=-3$, and $Q=1$, and I add the Quantity $nkkxx+2nklx+nll$, that is, $2xx-12x+18$ to each Part of the Æquation, and there comes out $x^4+2xx+1=2xx-12x+18$, and extracting on both Sides the Root $xx+1=x\sqrt{2}-3\sqrt{2}$. But if you had rather avoid the Extraction of the Root, make $xx+\frac{1}{2}px+Q=\sqrt{n}\times\overline{kx+l}$, and you'll find, as before, $xx+1=\pm\sqrt{2}\times\overline{x-3}$. And if again you extract the Root of this Æquation, there will come out $x=\pm\frac{1}{2}\sqrt{2}\pm\sqrt{\frac{-1}{2}\pm 3\sqrt{2}}$, that is, according to the

Variation of the Signs $x = \frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}$, and $x = \frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}}$. Alſo $x = -\frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}$, and $x = -\frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}}$. Which are four Roots of the Æquation at firſt propos'd, $x^4 + 12x - 17 = 0$. But the two laſt of them are impoſſible.

Let us now propoſe the Æquation $x^4 - 6x^3 - 58xx - 114x - 11 = 0$, and by writing $-6$, $-58$, $-114$, and $-11$, for $p$, $q$, $r$, and $s$ reſpectively, there will ariſe $-67 = \alpha$, $-315 = \beta$, and $-1133\frac{1}{4} = \zeta$; the only common Diviſor of the Numbers $\beta$ and $2\zeta$, or of $-315$ and $-\frac{4533}{2}$ is 3, and conſequently will be here $n$, and the Diviſors of $\frac{\beta}{n}$ or $-105$, are 3, 5, 7, 15, 21, 35, and 105, which are therefore to be try'd for $k$. Wherefore, I try firſt 3, and the Quotient $-35$ which (comes out by dividing $\frac{\beta}{n}$ by $k$, or $-105$ by 3) I ſubtract from $\frac{1}{2}pk$, or $-3 \times 3$, and there remains 26, the half whereof, 13, ought to be $l$. But $\frac{\alpha + nkk}{2}$, or $\frac{-67 + 27}{2}$, that is, $-20$, will be $Q$, and $QQ - s$ will be 411, which may be divided by $n$, or 3, but the Root of the Quotient, 137, cannot be extracted. Wherefore I reject 3, and try 5 for $k$. The Quotient that now comes out by dividing $\frac{\beta}{n}$ by $k$, or $-105$ by 5, is $-21$; and ſubtracting this from $\frac{1}{2}pk$, or $-3 \times 5$, there remains 6, the half whereof, 3, is $l$. Alſo $Q$, or $\frac{\alpha + nkk}{2}$, that is, $\frac{-67 + 75}{2}$, is the Number 4. And $QQ - s$, or $16 + 11$, may be divided by $n$; and the Root of the Quotient, which is 9, being extracted, *i. e.* 3 agrees with $l$. Wherefore I conclude, that $l$ is $= 3$, $k = 5$, $Q = 4$, and $n = 3$; and if $nkkxx + 2nklx + nll$, that is, $75xx + 90x + 27$, be added to each Part of the Æquation, the Root may be extracted on both Sides, and there will come out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, or $xx - 3x + 4 = \pm\sqrt{3} \times$

$\overline{5x+3}$; and the Root being again extracted, $x = \frac{3 \pm 5\sqrt{3}}{2}$

$\pm \sqrt{17 \pm \frac{21 \times \sqrt{3}}{2}}$.

Thus, if there was propos'd the Æquation $x^4 - 9x^3 + 15xx - 27x + 9 = 0$, by writing $-9$, $+15$, $-27$, and $+9$ for $p$, $q$, $r$, and $s$ reſpectively, there will come out $-5\frac{3}{4} = \alpha$, $-50\frac{5}{8} = \beta$, and $2\frac{7}{64} = \zeta$. The common Diviſors of $\beta$ and $2\zeta$, or $-\frac{405}{8}$ and $\frac{135}{32}$ are 3, 5, 9, 15, 27, 45, and 135; but 9 is a Square Number, and 3, 15, 27, 135, divided by the Number 4, do not leave Unity, as, by reaſon of the odd Term $p$ they ought to do. Theſe therefore being rejected, there remain only 5 and 45 to be try'd for $n$. Let us put therefore, firſt $n = 5$, and the odd Diviſors of $\frac{\beta}{n}$ or $-\frac{81}{8}$ being halv'd, *viz.* $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, $\frac{27}{2}$, $\frac{81}{2}$, are to be try'd for $k$. If $k$ be made $\frac{1}{2}$, the Quotient $-\frac{81}{2}$, which comes out by dividing $\frac{\beta}{n}$ by $k$, ſubtracted from $\frac{1}{2}pk$, or $-\frac{9}{4}$, leaves 18 for $l$, and $\frac{\alpha + nkk}{2}$, or $-2$, is $Q$, and $QQ - s$, or $-5$, may be divided by $n$, or 5; but the Root of the Negative Quotient $-1$ is impoſſible, which yet ought to be 18. Wherefore I conclude $k$ not to be $\frac{1}{2}$, and then I try if it be $\frac{3}{2}$. The Quotient which ariſes by dividing $\frac{\beta}{n}$ by $k$, or $-\frac{81}{8}$ by $\frac{3}{2}$, *viz.* the Quotient $-\frac{27}{4}$ I ſubtract from $\frac{1}{2}pk$, or $-\frac{27}{4}$, and there remains 0; whence now $l$ will be nothing. But $\frac{\alpha + nkk}{2}$, or 3, is equal to $Q$, and $QQ - s$ is nothing; whence again $l$, which is the Root of $QQ - s$, divided by $n$, is found to be nothing. Wherefore theſe Things thus agreeing, I conclude $n$ to be $= 5$, $k = \frac{1}{2}$, $l = 0$, and $Q = 3$; and therefore by adding to each Part of the Æquation propos'd, the Terms $nkkxx + 2nlkx + nll$, that is, $\frac{45}{4}xx$, and by extracting on both Sides the Square Root, there comes out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, that is, $xx - 4\frac{1}{2}x + 3 = \sqrt{5} \times \frac{3}{2}x$.

By the ſame Method, Literal Æquations are alſo reduc'd. As if there was $x^4 - 2ax^3 \begin{matrix} +2aa \\ -cc \end{matrix} xx - 2a^3x + a^4 = 0$.

by ſubſtituting $-2a$, $2aa-cc$, $-2a^3$, and $+a^4$ for $p$, $q$, $r$, and $s$ reſpectively, you obtain $aa-cc=\alpha$, $-acc-a^3=\beta$, and $\frac{3}{4}a^4+\frac{1}{2}aacc-\frac{1}{4}c^4=\zeta$. The common Diviſor of the Quantities $\beta$ and $2\zeta$ is $aa+cc$, which then will be $n$; and $\frac{\beta}{n}$ or $-a$, has the Diviſors 1 and $a$. But becauſe $n$ is of two Dimenſions, and $k\sqrt{n}$ ought to be of no more than one, therefore $k$ will be of none, and conſequently cannot be $a$. Let therefore $k$ be 1, and $\frac{\beta}{n}$ being divided by $k$, take the Quotient $-a$ from $\frac{1}{2}pk$, and there will remain nothing for $l$. Moreover, $\frac{\alpha+nkk}{2}$, or $aa$, is $Q$, and $QQ-s$, or $a^4-a^4$, is 0; and thence again there comes out nothing for $l$. Which ſhews the Quantities $n$, $k$, $l$, and $Q$, to be rightly found; and adding to each Part of the Æquation propos'd, the Terms $nkkxx+2nklx+nll$, that is, $aaxx+ccxx$, the Root may be extracted on both Sides; and by that Extraction there will come out $xx+\frac{1}{2}px+Q=\sqrt{n}\times\overline{kx+l}$, that is, $xx-ax+aa=\pm x\sqrt{aa+cc}$. And the Root being again extracted, you'll have $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{aa+cc}\pm\sqrt{\frac{1}{4}cc-\frac{1}{2}aa\pm\frac{1}{2}a\sqrt{aa+cc}}$.

Hitherto I have apply'd the Rule to the Extraction of Surd Roots; the ſame may alſo be apply'd to the Extraction of Rational Roots, if for the Quantity $n$ you make Uſe of Unity: and after that Manner we may examine, whether an Æquation that wants Fracted or Surd Terms can admit of any Diviſor, either Rational or Surd, of two Dimenſions. As if the Æquation $x^4-x^3-5xx+12x-6=0$ was propos'd, by ſubſtituting $-1$, $-5$, $+12$, and $-6$ for $p$, $q$, $r$, and $s$ reſpectively, you'll find $-5\frac{1}{4}=\alpha$, $9\frac{3}{8}=\beta$, and $-10\frac{57}{64}=\zeta$. The common Diviſor of the Terms $\beta$ and $2\zeta$, or of $\frac{75}{8}$ and $-\frac{697}{32}$ is only Unity. Wherefore I put $n=1$. The Diviſors of the Quantity $\frac{\beta}{n}$, or $\frac{75}{8}$, are 1, 3, 5, 15, 25, 75; the Halves whereof (if $p$ be odd) are to be try'd for $k$. And if for $k$ we try $\frac{5}{2}$, you'll have $\frac{1}{2}pk-\frac{\beta}{nk}=-5$, and its half $-\frac{5}{2}=l$. Alſo

$\alpha+nkk$

$\frac{\alpha + nkk}{2} = \frac{1}{2} = Q$, and $\frac{QQ - s}{n} = 6\frac{1}{4}$, the Root whereof agrees with $l$. I therefore conclude, that the Quantities $n$, $k$, $l$, $Q$ are rightly found; and having added to each Part of the Æquation the Terms $nkkxx + 2nklx + nll$, that is, $6\frac{1}{4}xx - 12\frac{1}{2}x + 6\frac{1}{4}$, the Root may be extracted on both Sides; and by that Extraction there will come out $xx + \frac{1}{2}px + Q = \pm\sqrt{n} \times \overline{kx + l}$, that is, $xx - \frac{1}{2}x + \frac{1}{2} = \pm 1 \times 2\frac{1}{2}x - 2\frac{1}{2}$, or $xx - 3x + 3 = 0$, and $xx + 2x - 2 = 0$, and ſo by theſe two Quadratick Æquations the Biquadratick one propos'd may be divided. But Rational Diviſors of this Sort may more expeditiouſly be found by the other Method deliver'd above.

If at any Time there are many Diviſors of the Quantity $\frac{\beta}{n}$, ſo that it may be too difficult to try all of them for $k$, their Number may be ſoon diminiſh'd, by ſeeking all the Diviſors of the Quantity, $\alpha s - \frac{1}{4}rr$. For the Quantity $Q$ ought to be equal to ſome one of theſe, or to the half of ſome odd one. Thus, in the laſt Example, $\alpha s - \frac{1}{4}rr$ is $-\frac{9}{2}$, ſome one of whoſe Diviſors, 1, 3, 9, or of them halv'd $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, ought to be $Q$. Wherefore, by trying ſingly the halv'd Diviſors of the Quantity $\frac{\beta}{n}$, *viz.* $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, and $\frac{75}{2}$ for $k$, I reject all that do not make $\frac{1}{2}\alpha + \frac{1}{2}nkk$, or $-\frac{21}{8} + \frac{1}{2}kk$; that is, $Q$ is one of the Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$. But by writing $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, &c. for $k$, there come out reſpectively $-\frac{5}{2}$, $-\frac{3}{2}$, $+\frac{1}{2}$, $+\frac{51}{2}$, &c. for $Q$; out of which only $-\frac{3}{2}$ and $\frac{1}{2}$ are found among the aforeſaid Numbers 1, 3, 9, $\frac{1}{2}$. $\frac{3}{2}$, $\frac{9}{2}$, and conſequently the reſt being rejected, either $k$ will be $= \frac{3}{2}$ and $Q = -\frac{3}{2}$, or $k = \frac{5}{2}$ and $Q = \frac{1}{2}$. Which two Caſes are examin'd. And ſo much of Æquations of four Dimenſions.

If an Æquation of ſix Dimenſions is to be reduc'd, let it be $x^6 + px^5 + qx^4 + rx^3 + sxx + tx + v = 0$, and make

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}p\alpha = \beta. \quad s - \tfrac{1}{2}p\beta = \gamma.$$
$$\gamma - \tfrac{1}{4}\alpha\alpha = \zeta. \quad t - \tfrac{1}{2}\alpha\beta = \eta. \quad v - \tfrac{1}{4}\beta\beta = \theta.$$
$$\zeta\theta - \tfrac{1}{4}\eta\eta = \lambda.$$

Then take for $n$ ſome common Integer Diviſor, that is not a Square, out of the Terms $2\zeta$, $\eta$, $2\theta$, and that likewiſe is not diviſible by a Square Number, and which alſo divided by the Number 4, ſhall leave Unity; if but any one of the Terms $p$, $r$, $t$ be odd. For $k$ take ſome Integer Diviſor of the Quantity $\frac{\lambda}{2nn}$ if $p$ be even; or the half of an odd Diviſor if $p$ be odd; or 0 if $\lambda$ be 0. For $Q$ [take] the Quantity $\frac{1}{2}\alpha + \frac{1}{2}nkk$. For $l$ ſome Diviſor of the Quantity $\frac{Qr - QQp - t}{n}$ if $Q$ be an Integer; or the half of an odd Diviſor, if $Q$ be a Fraction that has for its Denominator the Number 2; or 0, if the Dividual [or the Quantity] $\frac{Qr - QQp - t}{n}$ be nothing. And for $R$ the Quantity $\frac{1}{2}r - \frac{1}{2}Qp + \frac{1}{2}nkl$. Then try if $RR - v$ can be divided by $n$, and the Root of the Quotient extracted; and beſides, if that Root be equal as well to the Quantity $\frac{QR - \frac{1}{2}t}{nl}$ as to the Quantity $\frac{QQ + pR - nll - s}{2nk}$. If all theſe happen, call that Root $m$; and in room of the Æquation propos'd, write this, $x^3 + \frac{1}{2}pxx + Qx + r = \pm \sqrt{n} \times \overline{kxx + lx + m}$. For this Æquation, by ſquaring its Parts, and taking from both Sides the Terms on the Right-Hand, will produce the Æquation propos'd. Now if all theſe Things do not happen in the Caſe propos'd, the Reduction will be impoſſible, if it appears beforehand that the Æquation cannot be reduc'd by a rational Diviſor.

For Example, let there be propos'd the Æquation

$$x^6 - 2ax^5 + 2bbx^4 + 2abbx^3 \begin{array}{l} -2aabb \\ +2a^3b \\ -4ab^3 \end{array} xx \begin{array}{l} +3aab^4 \\ -a^4bb \end{array} = 0,$$

and by writing $-2a$, $+2bb$, $+2abb$, $-2aabb + 2a^3b - 4ab^3$, 0, and $3aab^4 - a^4bb$ for $p$, $q$, $r$, $s$, $t$, and $v$ reſpectively, there will come out $2bb - aa = \alpha$. $4abb - a^3 = \beta$. $2a^3b + 2aabb - 4ab^3 - a^4 = \gamma$. $-b^4 + 2a^3b + 3aabb - 4ab^3 - \frac{5}{4}a^4 = \zeta$. $\frac{1}{2}a^5 - a^3bb = \eta$, and $3aab^4 - a^4bb - \frac{1}{4}a^6 = \theta$. And the common Diviſor of the Terms $2\zeta$, $\eta$, and $2\theta$, is $aa - 2bb$, or $2bb - aa$, according as $aa$ or $2bb$ is the greater. But let $aa$ be greater than $2bb$, and $aa - 2bb$ will be $n$. For $n$ muſt always be Affirmative. Moreover, $\frac{\zeta}{n}$ is $-\frac{5}{4}aa + 2ab + \frac{1}{2}db$, $\frac{\eta}{n}$ is $\frac{1}{2}a^3$,

$\frac{1}{2}a^3$, and $\frac{\theta}{n}$ is $-\frac{1}{4}a^4+\frac{1}{2}aabb$, and conſequently $\frac{\zeta}{2n}\times\frac{\theta}{n}-\frac{\eta\eta}{4nn}$, or $\frac{\lambda}{2nn}$, is $\frac{1}{4}a^6-\frac{1}{4}a^5b+\frac{1}{4}a^4bb-\frac{1}{2}a^3b^3-\frac{1}{4}aab^4$, the Diviſors whereof are 1, $a$, $aa$; but becauſe $\sqrt{n}\times k$ cannot be of more than one Dimenſion, and the $\sqrt{n}$ is of one, therefore $k$ will be of none; and conſequently can only be a Number. Wherefore, rejecting $a$ and $aa$, there remains only 1 for $k$. Beſides, $\frac{1}{2}\alpha+\frac{1}{2}nkk$ gives 0 for $Q$, and $\frac{Qr-QQp-t}{-n}$ is alſo nothing; and conſequently $l$, which ought to be its Diviſor, will be nothing. Laſtly, $\frac{1}{2}r-\frac{1}{2}pQ+\frac{1}{2}nkl$ gives $abb$ for $R$. And $RR-v$ is $-2aab^4+a^4bb$, which may be divided by $n$, or $aa-2bb$, and the Root of the Quotient $aabb$ be extracted, and that Root taken Negatively, *viz.* $-ab$, is not unequal to the indefinite Quantity $\frac{QR-\frac{1}{2}t}{nl}$, or $\frac{0}{0}$, but equal to the definite Quantity $\frac{QQ+pR-nll-s}{2nk}$. Wherefore that Root $-ab$ will be $m$, and in the room of the Æquation propos'd, there may be writ $x^3-\frac{1}{2}pxx+Qx+R=\sqrt{n}\times\overline{kxx+lx+m}$, that is, $x^3-axx+abb=\sqrt{aa-2bb}\times\overline{xx-ab}$. The Truth of which Æquation you may prove by ſquaring the Parts of the Æquation found, and taking away the Terms on the Right Hand from both Sides. From that Operation will be produc'd the Æquation $x^6-2ax^5+2bbx^4+2abbx^3-2aabbxx+2a^3bxx-4ab^3xx+3aab^4-a^4bb=0$, which was to be reduc'd.

If the Æquation is of eight Dimenſions, let it be $x^8+px^7+qx^6+rx^5+sx^4+tx^3+vxx+wx+z=0$, and make $q-\frac{1}{4}pp=\alpha$. $r-\frac{1}{2}p\alpha=\beta$. $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$. $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$. $v-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$. $w-\frac{1}{2}\beta\gamma=\zeta$, and $z-\frac{1}{4}\gamma\gamma=\eta$. And ſeek a common Diviſor of the Terms $2\delta$, $2\varepsilon$, $2\zeta$, $8\eta$, that ſhall be an Integer, and neither a Square Number, nor diviſible by a Square Number; and which alſo divided by 4 ſhall leave Unity, if any of the alternate Terms $p$, $r$, $t$, $w$ be odd. If there be no ſuch common Diviſor, it is certain, that the Æquation cannot be reduc'd by the Extraction of a Quadratick Surd Root,

and

and if it cannot be ſo reduc'd, there will ſcarce be found a common Diviſor of all thoſe four Quantities. The Operation therefore hitherto is a Sort of an Examination, whether the Æquation be reducible or not; and conſequently, ſince that Sort of Reductions are ſeldom poſſible, it will moſt commonly end the Work.

And, by a like Reaſon, if the Æquation be of ten, twelve, or more Dimenſions, the Impoſſibility of its Reduction may be known. As if it be $x^{10} + px^9 + qx^8 + rx^7 + sx^6 + tx^5 + vx^4 + ax^3 + bx^2 + cx + d = 0$, you muſt make $q - \frac{1}{4}pp = \alpha$, $r - \frac{1}{2}p\alpha = \beta$, $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$, $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = \delta$, $v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \varepsilon$, $a - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma = \zeta$, $b - \frac{1}{2}\beta\delta - \frac{1}{4}\gamma\gamma = \eta$, $c - \frac{1}{2}\gamma\delta = \theta$, $d - \frac{1}{4}\delta\delta = \varkappa$. And ſeek ſuch a common Diviſor to the five Terms, $2\varepsilon$, $2\zeta$, $8\eta$, $4\theta$, $8\varkappa$, as is an Integer, and not a Square, but which ſhall leave 1 when divided by 4, if any one of the Terms $p$, $r$, $t$, $a$, $c$ be odd.

So if there be an Æquation of twelve Dimenſions, as $x^{12} + px^{11} + qx^{10} + rx^9 + sx^8 + tx^7 + vx^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, make $q - \frac{1}{4}pp = \alpha$, $r - \frac{1}{2}p\alpha = \beta$, $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$, $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = \delta$, $v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \varepsilon$, $a - \frac{1}{2}p\varepsilon - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma = \zeta$, $b - \frac{1}{2}\alpha\varepsilon - \frac{1}{2}\beta\delta - \frac{1}{4}\gamma\gamma = \eta$, $c - \frac{1}{2}\beta\varepsilon - \frac{1}{2}\gamma\delta = \theta$, $d - \frac{1}{2}\gamma\varepsilon - \frac{1}{4}\delta\delta = \varkappa$, $e - \frac{1}{2}\delta\varepsilon = \lambda$, $f - \frac{1}{4}\varepsilon\varepsilon = \mu$, and you muſt ſeek a common Integer Diviſor of the ſix Terms $2\zeta$, $8\eta$, $4\theta$, $8\varkappa$, $4\lambda$, $8\mu$, that is not a Square, but being divided by 4 ſhall leave Unity, if any one of the Terms $p$, $r$, $t$, $a$, $c$, $e$ be odd.

And thus you may go on *ad infinitum*, and the propos'd Æquation will be always irreduceable when it has no common Diviſor. But if at any Time ſuch a Diviſor $n$ being found, there are Hopes of a future Reduction, and it may be found by working or following the Steps of the Operation we ſhew'd in the Æquation of eight Dimenſions.

Seek a Square Number, to which after it is multiply'd by $n$, the laſt Term $z$ of the Æquation being added under its proper Sign, ſhall make a Square Number. But that may be expeditiouſly perform'd if you add to $z$, where $n$ is an even Number, or to $4z$ when it is odd, theſe Quantities ſucceſſively $n$, $3n$, $5n$, $7n$, $9n$, $11n$, and ſo on till the Sum becomes equal to ſome Number in the Table of Square

Numbers, which I ſuppoſe to be ready at Hand. And if no ſuch Square Number occurs before the Square Root of that Sum, augmented by the Square Root of the Exceſs of that Sum above the laſt Term of the Æquation, is four times greater than the greateſt of the Terms of the propos'd Æquation $p, q, r, s, t, v$, &c. there will be no Occaſion to try any farther. For then the Æquation cannot be reduc'd. But if ſuch a Square Number does accordingly occur, let its Root be $S$ if $n$ is even, or $2S$ if $n$ be odd; and call the $\sqrt{\frac{SS-z}{n}}=h$. But $s$ and $h$ ought to be Integers if $n$ is even, but if $n$ is odd, they may be Fractions that have 2 for their Denominator. And if one is a Fraction, the other ought to be ſo too. Which alſo is to be obſerv'd of the Numbers $R$ and $M$, $Q$ and $l$, $P$ and $k$ hereafter to be found. And all the Numbers $S$ and $h$, that can be found within the preſcrib'd Limit, muſt be collected in a [Table or] Catalogue.

Afterwards, for ($k$) all the Numbers are to be ſucceſſively try'd, which do not make $nk\pm\frac{1}{2}p$ four times greater than the greateſt Term of the Æquation, and you muſt in all Caſes put $\frac{nkk+\alpha}{2}=Q$. Then you are to try ſucceſſively for $l$ all the Numbers that do not make $nl\pm Q$ four times greater than the greateſt Term of the Æquation; and in every Tryal put $\frac{-npkk+2\beta}{4}+nkl=R$. Laſtly, for $m$ you muſt try ſucceſſively all the Numbers which do not make $nm+R$ four times greater than the greateſt of the Terms of the Æquation, and you muſt ſee whether in any Caſe if you make $s-QQ-PR+nll=2H$, and $H+nkm=S$, let $S$ be ſome of the Numbers which were before brought into the Catalogue for $S$; and beſides, if the other Number anſwering to that $S$, which being ſet down for $h$ in the ſame Catalogue, will be equal to theſe three, $\frac{2RS-w}{2nm}$, $\frac{2QS+RR-v-nmm}{2nl}$, and $\frac{PS+2QR-t-2nlm}{2nk}$. If all theſe Things ſhall happen in any Caſe, inſtead of the Æquation propos'd, you muſt write this $x^4+\frac{1}{2}px^3+Qxx+Rx+S=\sqrt{n}\times kx^3+lxx+mx+h$.

For Example, let there be propos'd the Æquation $x^8 + 4x^7 - x^5 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x - 5 = 0$, and you'll have $q - \frac{1}{4}pp = -1 - 4 = -5 = \alpha$. $r - \frac{1}{2}p\alpha = -10 + 10 = 0 = \beta$. $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = 5 - \frac{25}{4} = -\frac{5}{4} = \gamma$. $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = -5 + \frac{5}{2} = -\frac{5}{2} = \delta$. $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = -10 - \frac{25}{8} = -\frac{105}{8}$. $w - \frac{1}{2}\beta\gamma = -10 = \zeta$. $z - \frac{1}{4}\gamma\gamma = -5 - \frac{25}{6} = -\frac{345}{6} = \eta$. Therefore $2\delta$, $2\varepsilon$, $2\zeta$, $8\eta$ reſpectively are $-5$; $-\frac{105}{4}$, $-20$, and $-\frac{345}{8}$, and their common Diviſor 5. which divided by 4, leaves 1, as it ought, becauſe the Term $s$ is odd. Since therefore the common Diviſor $n$, or 5, is found, which gives hope to a future Reduction, and becauſe it is odd to $4z$, or $-20$, I ſucceſſively add $n$, $3n$, $5n$, $7n$, $9n$, &c. or 5, 15, 25, 35, 45, &c. and there ariſes $-15$, 0, 25, 60, 105, 160, 225, 300, 385, 480, 585, 700, 825, 960, 1105, 1260, 1425, 1600. Of which only 0, 25, 225, and 1600 are Squares. And the Halves of theſe Roots 0, $\frac{5}{2}$, $\frac{15}{2}$, 20, collect in a Table for the Values of $S$, and ſo the Values of $\sqrt{\frac{SS-z}{n}}$, that is, 1, $\frac{3}{2}$, $\frac{7}{2}$, 9, for $b$. But becauſe $S + nb$, if 20 be taken for $S$ and 9 for $b$, becomes 65, a Number greater than four times the greateſt Term of the Æquation, therefore I reject 20 and 9, and write only the reſt in the Table as follows:

| | | | |
|---|---|---|---|
| $b$ | 1. | $\frac{3}{2}$. | $\frac{7}{2}$. |
| $S$ | 0. | $\frac{5}{2}$. | $\frac{15}{2}$. |

Then try for $k$ all the Numbers which do not make $\frac{1}{2} \pm nk$, or $2 \pm 5k$, greater than 40, (four times the greateſt Term of the Æquation) that is, the Numbers $-8$, $-7$, $-6$, $-5$, $-4$, $-3$, $-2$, $-1$, 0, 1, 2, 3, 4, 5, 6, 7, putting $\frac{nkk+\alpha}{2}$, or $\frac{5kk-5}{2}$, that is, the Numbers $\frac{315}{2}$, 120, $\frac{175}{2}$, 60, $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, 60, $\frac{175}{2}$, 120, reſpectively for $Q$. But even when $Q \pm nl$, and much more $Q$, ought not to be greater than 40, I perceive I am to reject $\frac{315}{2}$, 120, $\frac{175}{2}$, and 60, and their Correſpondents $-8$, $-7$, $-6$, $-5$, 5, 6, 7, and conſequently that only $-4$, $-3$, $-2$, $-1$, 0, 1, 2, 3, 4, muſt reſpectively be try'd for $k$, and $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, reſpectively for $Q$. Let us therefore try $-1$ for $k$, and 0 for $Q$, and in this Caſe for $l$ there will be ſucceſſively to be try'd all the Numbers which do not make $Q \pm nl$ greater than

40, that is, all the Numbers between 10 and $-10$; and for $R$ you are reſpectively to try the Numbers $\frac{2\beta - npkk}{4} + nkl$, or $-5 - 5l$, that is, $-55$, $-50$, $-45$, $-40$, $-35$, $-30$, $-25$, $-20$, $-15$, $-10$, $-5$, 0, 5, 10, 15, 20, 25, 35, 40, 45, the three former of which and the laſt, becauſe they are greater than 40, may be neglected. Let us try therefore $-2$ for $l$, and 5 for $R$, and in this Caſe for $m$ there will be beſides to be try'd all the Numbers which do not make $R \pm mn$, or $5 \pm mn$, greater than 40, that is, all the Numbers between 7 and $-9$, and ſee whether or not by putting $s - QQ - pR + nll$, that is $5 - 20 + 20$, or $5 = 2H$, let $H + nkm$, or $\frac{5}{2} - 5m = S$, that is, if any of theſe Numbers $\frac{-65}{2}$, $\frac{-55}{2}$, $\frac{-45}{2}$, $\frac{-35}{2}$, $\frac{-25}{2}$, $\frac{-15}{2}$, $\frac{-5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, $\frac{35}{2}$, $\frac{45}{2}$, $\frac{55}{2}$, $\frac{65}{2}$, $\frac{75}{2}$, $\frac{85}{2}$, is equal to any of the Numbers 0, $\pm\frac{5}{2}$, $\pm\frac{15}{2}$, which were firſt brought into the Catalogue for $S$. And we meet with four of theſe $-\frac{15}{2}$, $-\frac{5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, to which anſwer $\pm\frac{7}{2}$, $\pm\frac{3}{2}$, $\pm\frac{3}{2}$, $\pm\frac{7}{2}$, being writ for $h$ in the ſame Table, as alſo 2, 1, 0, $-1$ ſubſtituted for $m$. But let us try $-\frac{5}{2}$ for $S$, 1 for $m$, and $\pm\frac{3}{2}$ for $h$, and you'll have $\frac{2RS - w}{2nm} = \frac{-25 + 10}{10} = -\frac{3}{2}$, and $\frac{2QS + RR - Vnmm}{2nl} = \frac{25 + 10 - 5}{-20} = -\frac{3}{2}$, and $\frac{pS + 2QR - t - 2nlm}{2nk} = \frac{-10 + 5 + 20}{-10} = -\frac{3}{2}$. Wherefore, ſince there comes out in all Caſes $-\frac{3}{2}$, or $h$, I conclude all the Numbers to be rightly found, and conſequently that in room of the Æquation propos'd, you muſt write $x^4 + \frac{1}{2}px^3 + Qxx + Rx + S = \sqrt{n} \times \overline{kx^3 + lxx + mx + h}$, that is, $x^4 + 2x^3 + 5x - 2\frac{1}{2} = \sqrt{5} \times \overline{-x^3 - 2xx + x - 1\frac{1}{2}}$. For by ſquaring the Parts of this, there will be produc'd that Æquation of eight Dimenſions, which was at firſt propos'd.

Now, if by trying all the Caſes of the Numbers, all the aforeſaid Values of $h$ do not in any Caſe conſent, it would be an Argument that the Æquation could not be ſolv'd by the Extraction of the Surd Quadratick Root.

I might now join the Reductions of Æquations by the Extraction of the Surd Cubick Root, but theſe, as being ſeldom of Uſe, I paſs by. Yet there are ſome Reductions of Cubick Æquations commonly known, which, if I ſhould wholly paſs over, the Reader might perhaps think us deficient. Let there be propos'd the Cubick Æquation $x^3 * + qx + r = 0$; the ſecond Term whereof is wanting: For that every Cubick Æquation may be reduc'd to this Form, is evident from what we have ſaid above. Let $x$ be ſuppos'd $= a + b$. Then will $a^3 + 3aab + 3abb + b^3$ (that is $x^3$) $+ qx + r = 0$. Let $3aab + 3abb$ (that is, $3abx$) $+ qx = 0$, and then will $a^3 + b^3 + r = 0$. By the former Æquation $b$ is $= -\frac{q}{3a}$, and cubically $b^3 = -\frac{q^3}{27a^3}$. Therefore by the lattter, $a^3 - \frac{q^3}{27a^3} + r = 0$, or $a^6 + ra^3 = \frac{q^3}{27}$, and by the Extraction of the adfected Quadratick Root, $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Extract the Cubick Root and you'll have $a$. And above, you had $\frac{-q}{3a} = b$, and $a + b = x$. Therefore $a - \frac{q}{3a}$ is the Root of the Æquation propos'd.

For Example, let there be propos'd the Æquation $y^3 - 6yy + 6y + 12 = 0$. To take away the ſecond Term of this Æquation, make $x + 2 = y$, and there will ariſe $x^3 * - 6x + 8 = 0$. Where $q$ is $= -6$, $r = 8$, $\frac{1}{4}rr = 16$, $\frac{q^3}{27} = -8$, $a^3 = -4 \pm \sqrt{8}$, $a - \frac{q}{3a} = x$, and $x + 2 = y$, that is, $2 + \sqrt[3]{-4 \pm \sqrt{8}} + \frac{2}{\sqrt[3]{-4 \pm \sqrt{8}}}$.

And after this Way the Roots of all Cubical Æquations may be extracted wherein $q$ is Affirmative; or alſo wherein $q$ is Negative, and $\frac{q^3}{27}$ not greater than $\frac{1}{4}rr$, that is, wherein two of the Roots of the Æquation are impoſſible. But where $q$ is Negative, and $\frac{q^3}{27}$ at the ſame time greater than $\frac{1}{4}rr$, $\sqrt{\frac{1}{4}rr - \frac{q^3}{27}}$ becomes an impoſſible Quantity, and ſo the Root of the Æquation $x$ or $y$ will, in this Caſe, be impoſſible, *viz.* in this Caſe there are three poſſible Roots, which all of them are alike with reſpect to the Terms of the Æquations $q$ and $r$, and are indifferently denoted by the Letters $x$ and $y$, and conſequently all of them may be extracted by the ſame Method, and expreſs'd the ſame Way as any one is extracted or expreſs'd; but it is impoſſible to expreſs all three by the Law aforeſaid. The Quantity $a - \frac{q}{3a}$, whereby $x$ is denoted, cannot be manyfold, and for that Reaſon the Suppoſition that $x$, in this Caſe wherein it is triple, may be equal to the Binomial $a - \frac{q}{3a}$, or $a + b$, the Cubes of whoſe Terms $a^3 + b^3$ are together $= r$, and the triple Rectangle $3ab$ is $= q$, is plainly impoſſible; and it is no Wonder that from an impoſſible Hypotheſis, an impoſſible Concluſion ſhould follow.

There is, moreover, another Way of expreſſing theſe Roots, *viz.* from $a^3 \overline{+ b^3 + r}$, that is, from nothing take $a^3 + r$, or $\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$, and there will remain $b^3 = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Therefore $a$ is $=$

$\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and $b =$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$; or $a =$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and $b =$

$\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and conſequently the Sum of

theſe $\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}} +$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$ will be $x$.

Moreover, the Roots of Biquadratick Æquations may be extracted and expreſs'd by means of Cubick ones. But firſt you muſt take away the ſecond Term of the Æquation. Let the Æquation that [then] reſults be $x^4+qxx+rx+s=0$. Suppoſe this to be generated by the Multiplication of theſe two $xx+ex+f=0$, and $xx-ex+g=0$, that is, to be the ſame with this $x^4 * \begin{matrix}+f\\+g\\-ee\end{matrix}\, xx \begin{matrix}+eg\\-ef\end{matrix}\, x + fg=0$, and comparing the Terms you'll have $f+g-ee=q$, $eg-ef=r$, and $fg=s$. Wherefore $q+ee=f+g$, $\frac{r}{e}=g-f$, $\frac{q+ee+\frac{r}{e}}{2}=g$, $\frac{q+ee-\frac{r}{e}}{2}=f$, $\frac{qq+2eeq+e^4-\frac{rr}{ee}}{4}(=fg)=s$, and by Reduction $e^6+2qe^4\begin{matrix}+qq\\-4s\end{matrix}\,ee-rr=0$. For $ee$ write $y$, and you'll have $y^3+2qyy\begin{matrix}+qq\\-4s\end{matrix}\,y-rr=0$, a Cubick Æquation, whoſe ſecond Term may be taken away, and then the Root extracted either by the precedent Rule or otherwiſe. Then that Root being had, you muſt go back again, by putting $\sqrt{y}=e$, $\frac{q+ee-\frac{r}{e}}{2}=f$, $\frac{q+ee+\frac{r}{e}}{2}=g$, and the two Æquations $xx+ex+f=0$, and $xx-ex+g=0$, their Roots being extracted, will give the four Roots of the Biquadratick Æquation $x^4+qxx+rx+s=0$, *viz.* $x=-\frac{1}{2}e\pm\sqrt{\frac{1}{4}ee-f}$, and $x=\frac{1}{2}e\pm\sqrt{\frac{1}{4}ee-g}$. Where note, that if the four Roots of the Biquadratick Æquation are poſſible, the three Roots of the Cubick Æquation $y^3+2qyy\begin{matrix}+qq\\-4s\end{matrix}\,y-rr=0$ will be poſſible alſo, and conſequently cannot be extracted by the precedent Rule. And thus, if the affected Roots of an Æquation of five or more Dimenſions are converted into Roots that are not affected, the middle Terms of the Æquation being taken away, that Expreſſion of the Roots will be always impoſſible, where more than one Root in an Æquation of odd Dimenſions are poſſible, or more than two in an Æquation of even Dimenſions, which cannot be reduc'd by the Extraction of the Surd Quadratick Root, by the Method laid down above.

Monſieur *Des Cartes* taught how to reduce a Biquadratick Æquation by the Rules laſt deliver'd. *E.g.* Let there be propos'd the Æquation reduc'd above, $x^4 - x^3 - 5xx + 12x - 6 = 0$. Take away the ſecond Term, by writing $v + \frac{1}{4}$ for $x$, and there will ariſe $v^4 - \frac{43}{8}vv + \frac{75}{8}v - \frac{851}{256} = 0$. To take away the Fractions, write $\frac{1}{4}z$ for $v$, and there will ariſe $z^4 - 86zz + 600z - 851 = 0$. Here $-86 = q$, $600 = r$, and $-851 = s$ and conſequently $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$, and ſubſtituting what is equivalent, you'll have $y^3 - 172yy + 10800y - 360000 = 0$. Where trying all the Diviſors of the laſt Term 1, $-1$, 2, $-2$, 3, $-3$, 4, $-4$, 5, $-5$, and ſo onwards to 100, you'll find at length $y = 100$. Which yet may be found far more expeditiouſly by our Method above deliver'd. Then having got $y$, its Root 10 will be $e$, and $\dfrac{q + ee - \frac{r}{e}}{2}$, that is, $\dfrac{-86 + 100 - 60}{2}$, or $-23$, will be $f$, and $\dfrac{q + ee + \frac{r}{e}}{2}$, or 37 will be $g$, and conſequently the Æquations $xx + ex + f = 0$, and $xx - ex + g = 0$, and writing $z$ for $x$, and ſubſtituting equivalent Quantities, will become $zz + 10z - 23 = 0$, and $zz - 10z + 37 = 0$. Reſtore $v$ in the room of $\frac{1}{4}z$. and there will ariſe $vv + 2\frac{1}{2}v - \frac{23}{16} = 0$, and $vv - 2\frac{1}{2}v + \frac{37}{16} = 0$. Reſtore, moreover, $x - \frac{1}{4}$ for $v$, and there will come out $xx + 2x - 2 = 0$, and $xx - 3x + 3 = 0$, two Æquations; the four Roots whereof $x = -1 \pm \sqrt{3}$, and $x = 1\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$, are the ſame with the four Roots of the Biquadratick Æquation propos'd at the Beginning, $x^4 - x^3 - 5xx + 12x - 6 = 0$. But theſe might have been more eaſily found by the Method of finding Diviſors, explain'd before.

Hitherto

Hitherto it will ſuffice, I ſuppoſe, to have given the Reductions of Æquations after a more eaſy and more general Way than what has been done by others. But ſince among theſe Operations we often meet with complex radical Quantities, which may be reduc'd to more ſimple ones, it is convenient to explain the Reduction of thoſe alſo. They are perform'd by the Extractions of Roots out of Binomial Quantities, or out of Quantities more compounded, which may be conſider'd as Binomial ones.

*The Extraction of Roots out of Binomial Quantities.*

[But ſince this is already done in the Chapter of the *Reduction of Radicals to more ſimple Radicals, by means of the Extraction of Roots*, we ſhall ſay no more of it here.]

# THE
# Linear Conſtruction
# OF
# ÆQUATIONS.

HITHERTO I have ſhewn the Properties, Tranſmutations, Limits, and Reductions of all Sorts of Æquations. I have not always joyn'd the Demonſtrations, becauſe they ſeem'd too eaſy to need it, and ſometimes cannot be laid down without too much Tediouſneſs. It remains now only to ſhew, how, after Æquations are reduc'd to their moſt commodious Form, their Roots may be extracted in Numbers. And here the chief Difficulty lies in obtaining the two or three firſt Figures; which may be moſt commodiouſly done by either the Geometrical or Mechanical Conſtruction of an Æquation. Wherefore I ſhall ſubjoin ſome of theſe Conſtructions.

The Antients, as we learn from *Pappus*, in vain endeavour'd at the Triſection of an Angle, and the finding out of two mean Proportionals by a right Line and a Circle. Afterwards they began to conſider the Properties of ſeveral other Lines, as the Conchoid, the Ciſſoid, and the Conick Sections, and by ſome of theſe to ſolve thoſe Problems. At length, having more throughly examin'd the Matter, and the Conick Sections being receiv'd into Geometry, they diſtinguiſh'd Problems into three Kinds, *viz.* (1.) Into Plane ones, which deriving their Original from Lines on a Plane, may be ſolv'd by a right Line and a Circle; (2.) Into Solid ones, which were ſolved by Lines deriving their Original from the Conſideration of a Solid, that is, of a Cone; (3.) And Linear ones, to the Solution of which were requir'd Lines more compounded. And according to this Diſtinction, we are not to ſolve ſolid Problems by other Lines than the Conick Sections; eſpecially if no other Lines but right ones, a Circle, and the Conick Sections, muſt be receiv'd into Geometry. But the Moderns advancing yet much farther, have receiv'd into Geometry all Lines that can be expreſs'd by Æquations and have diſtinguiſh'd, according to the Dimenſions of the Æquations, thoſe Lines into Kinds; and have made it a Law, that you are not to conſtruct a Problem by a Line of a ſuperior Kind, that may be conſtructed by one of an inferior one. In the Contemplation of Lines, and finding out their Properties, I like their Diſtinction of them into Kinds, according to the Dimenſions of the Æquations by which they are defin'd. But it is not the Æquation, but the Deſcription that makes the Curve to be a Geometrical one. The Circle is a Geometrical Line, not becauſe it may be expreſs'd by an Æquation, but becauſe its Deſcription is a Poſtulate. It is not the Simplicity of the Æquation, but the Eaſineſs of the Deſcription, which is to determine the Choice of our Lines for the Conſtruction of Problems. For the Æquation that expreſſes a Parabola, is more ſimple than That that expreſſes a Circle, and yet the Circle, by reaſon of its more ſimple Conſtruction, is admitted before it. The Circle and the Conick Sections, if you regard the Dimenſion of the Æquations, are of the ſame Order, and yet the Circle is not number'd with them in the Conſtruction of Problems, but by reaſon of its ſimple Deſcription, is depreſs'd to a lower Order, *viz.* that of a right Line; ſo that it is not improper to expreſs that by a Circle that may be expreſs'd by a right Line. But it is a Fault to conſtruct that by the Conick Sections which may be conſtructed by a Circle. Either therefore you muſt take your Law and Rule from the Dimenſions of Æquations as obſerv'd in a Circle, and ſo take away the Diſtinction between Plane and Solid Problems; or elſe you muſt grant, that that Law is not ſo ſtrictly to be obſerv'd in Lines of ſuperior Kinds, but that ſome, by reaſon of their more ſimple Deſcription, may be preferr'd to others of the ſame Order, and may be number'd with Lines of inferior Orders in the Conſtruction of Problems. In Conſtructions that are equally Geometrical, the moſt ſimple are always to be preferr'd. This Law is ſo univerſal, as to be without Exception. But Algebraick Expreſſions add nothing to the Simplicity of the Conſtruction; the bare Deſcriptions of the Lines only are here to be conſider'd; and theſe alone were conſider'd by thoſe Geometricians who joyn'd a Circle with a right Line. And as theſe are eaſy or hard, the Conſtruction becomes eaſy or hard: And therefore it is foreign to the Nature of the Thing, from any Thing elſe to eſtabliſh Laws about Conſtructions. Either therefore let us, with the Antients, exclude all Lines beſides the Circle, and perhaps the Conick Sections, out of Geometry, or admit all, according to the Simplicity of the Deſcription. If the Trochoid were admitted into Geometry, we might, by its Means, divide an Angle in any given Ratio. Would you therefore blame thoſe who ſhould make Uſe of this Line to divide an Angle in the Ratio of one Number to another, and contend that this Line was not defin'd by an Æquation, but that you muſt make Uſe of ſuch Lines as are defin'd by Æquations? If therefore, when an Angle was to be divided, for Inſtance, into 10001 Parts, we ſhould be oblig'd to bring a Curve defin'd by an Æquation of above an hundred Dimenſions to do the Buſineſs; which no Mortal could deſcribe, much leſs underſtand; and ſhould prefer this to the Trochoid, which is a Line well known, and deſcrib'd eaſily by the Motion of a Wheel or a Circle, who would not ſee the Abſurdity? Either therefore the Trochoid is not to be admitted at all into Geometry, or elſe, in the Conſtruction of Problems, it is to be preferr'd to all Lines of a more difficult Deſcription. And there is the ſame Reaſon for other Curves. For which Reaſon we approve of the Triſections of an Angle by a Conchoid, which *Archimedes* in his Lemma's, and *Pappus* in his Collections, have preferr'd to the Inventions of all others in this Caſe; becauſe we ought either to exclude all Lines, beſides the Circle and right Line, out of Geometry, or admit them according to the Simplicity of their Deſcriptions, in which Caſe the Conchoid yields to none, except the Circle. Æquations are Expreſſions of Arithmetical Computation, and properly have no Place in Geometry, except as far as Quantities truly Geometrical (that is, Lines, Surfaces, Solids, and Proportions) may be ſaid to be ſome equal to others. Multiplications, Diviſions, and ſuch ſort of Computations, are newly receiv'd into Geometry, and that unwarily, and contrary to the firſt Deſign of this Science. For whoſoever conſiders the Conſtruction of Problems by a right Line and a Circle, found out by the firſt

Geometricians, will eaſily perceive that Geometry was invented that we might expeditiouſly avoid, by drawing Lines, the Tediouſneſs of Computation. Therefore theſe two Sciences ought not to be confounded. The Antients did ſo induſtriouſly diſtinguiſh them from one another, that they never introduc'd Arithmetical Terms into Geometry. And the Moderns, by confounding both, have loſt the Simplicity in which all the Elegancy of Geometry conſiſts. Wherefore that is Arithmetically more ſimple which is determin'd by the more ſimple Æquations, but that is Geometrically more ſimple which is determin'd by the more ſimple drawing of Lines; and in Geometry, that ought to be reckon'd beſt which is Geometrically moſt ſimple. Wherefore, I ought not to be blamed, if, with that Prince of Mathematicians, *Archimedes*, and other Antients, I make Uſe of the Conchoid for the Conſtruction of ſolid Problems. But if any one thinks otherwiſe, let him know, that I am here ſolicitous not for a Geometrical Conſtruction but any one whatever, by which I may the neareſt Way find the Root of the Æquation in Numbers. For the ſake whereof I here premiſe this Lemmatical Problem.

## *To place the right Line BC of a given Length, ſo between two other given Lines AB, AC, that being produc'd, it ſhall paſs through the given Point P.*

IF the Line *BC* turn about the Pole *P*, and at the ſame time moves on its End *C* upon the right Line *AC*, its other End *B* ſhall deſcribe the Conchoid of the Antients. Let this cut the Line *AB* in the Point *B*. Join *PB*, and its Part *BC* will be the right Line which was to be drawn. And, by the ſame Law, the Line *BC* may be drawn where, inſtead of *AC*, ſome Curve Line is made Uſe of. [*Vide Figure* 90]

If any do not like this Conſtruction by a Conchoid, another, done by a Conick Section, may be ſubſtituted in its room. From the Point *P* to the right Line *AD*, *AE*, draw *PD*, *PE*, making the Parallelogram *EADP*, and from the Points *C* and *D* to the right Lines *AB* let fall the Perpendiculars *CF*, *DG*, as alſo from the Point *E* to the right

Line $AC$, produc'd towards $A$, let fall the Perpendicular $EH$, and making $AD=a$, $PD=b$, $BC=c$, $AG=d$, $AB=x$, and $AC=y$, you'll have $AD:AG::AC:AF$, and conſequently $AF=\frac{dy}{a}$. Moreover, you'll have $AB:AC::PD:CD$, or $x:y::b:a-y$. Therefore $by=ax-yx$, which is an Æquation expreſſive of an Hyperbola: And again, by the 13th of the 2d *Elem.* $BCq$ will be $=ACq+ABq-2FAB$, that is, $cc=yy+xx-\frac{2dxy}{a}$. Both Sides of the former Æquation being multiply'd by $\frac{2d}{a}$, take them from both Sides of this, and there will remain $cc-\frac{2bdy}{a}=yy+xx-2dx$, an Æquation expreſſing a Circle, where $x$ and $y$ are at right Angles. Wherefore, if you make theſe two Lines an Hyperbola and a Circle, by the Help of theſe Æquations, by their Interſection you'll have $x$ and $y$, or $AB$ and $AC$, which determine the Poſition of the right Line $BC$. But thoſe right Lines will be compounded after this Way.

Draw any two right Lines, $KL$ equal to $AD$, and $KM$ equal to $PD$, containing the right Angle $MKL$. Compleat the Parallelogram $KLMN$, and with the Aſymptotes $LN$, $MN$, deſcribe through the Point $K$ the Hyperbola $IKX$.

On $KM$ produc'd towards $K$, take $KP$ equal to $AG$, and $KQ$ equal to $BC$. And on $KL$ produc'd towards $K$, take $KR$ equal to $AH$, and $RS$ equal to $RQ$. Compleat the Parallelogram $PKRT$, and from the Center $T$, at the Interval $TS$, deſcribe a Circle. Let that cut the Hyperbola in the Point $X$. Let fall to $KP$ the Perpendicular $XY$, and $XY$ will be equal to $AC$, and $KY$ equal to $AB$. Which two Lines, $AC$ and $AB$, or one of them, with the Point $P$, determine the Poſition ſought of the right Line $BC$. To demonſtrate which Conſtruction, and its Caſes, according to the [different] Caſes of the Problem, I ſhall not here inſiſt. [*Vide Figure* 91.]

I ſay, by this Conſtruction, if you think fit, you may ſolve the Problem. But this Solution is too compounded to ſerve for any [particular] Uſes. It is only a Speculation, and Geometrical Speculations have juſt as much Elegancy as

Simplicity, and deſerve juſt ſo much Praiſe as they can promiſe Uſe. For which Reaſon, I prefer the Conchoid, as much the ſimpler, and not leſs Geometrical; and which is of eſpecial Uſe in the Reſolution of Æquations as by us propos'd. Premiſing therefore the preceding Lemma, we Geometrically conſtruct Cubick and Biquadratick Problems [*as which may be reduc'd to Cubick ones*] as follows. [*Vide Figures 92 and 93.*]

Let there be propos'd the Cubick Æquation $x^3 * + qx + r = 0$, whoſe ſecond Term is wanting, but the third is denoted under its Sign $+q$, and the fourth by $+r$. Draw any right Line, $KA$, which call $n$. On $KA$, produc'd on both Sides, take $KB = \frac{q}{n}$ to the ſame Side as $KA$, if $q$ be poſitive, otherwiſe to the contrary Part. Biſect $BA$ in $C$, and on $K$, as a Center with the Radius $KC$, deſcribe the Circle $CX$, and in it accommodate the right Line $CX$ equal to $\frac{r}{nn}$, producing it each Way. Join $AX$, which produce alſo both Ways; then between the Lines $CX$ and $AX$ inſcribe $EY$ of the ſame Length as $CA$, and which being produc'd, may paſs through the Point $K$; then ſhall $XY$ be the Root of the Æquation. [*Vide Figure 94.*] And of theſe Roots, thoſe will be Affirmative which fall from $X$ towards $C$, and thoſe Negative which fall on the contrary Side, if it be $+r$, but contrarily if it be $-r$.

## *Demonſtration.*

To demonſtrate which, I premiſe theſe Lemma's.

Lemma I. $YX : AK :: CX : KE$. Draw $KF$ parallel to $CX$; then becauſe of the ſimilar Triangles $ACX$, $AKF$, and $EYX$, $EKF$, there is $AC : AK :: CX : KF$, and $YX : YE$, or $AC :: KF : KE$; and therefore by Equality $YX : AK :: CX : KE$. Q. E. D.

Lemma II. $YX : AK :: CY : \overline{AK + KE}$. For by Compoſition of Proportion $YX : AK :: YX + CX$ (i. e. $CY$) $: \overline{AK + KE}$. Q. E. D.

**Lemma III.** $KE - BK : YX :: YX : AK$. For (by 12. *Elem.* 2.) $YKq - CKq = CYq - CY \times CX = CY \times YX$. That is, if the Theorem be reſolv'd into Proportionals, $CY : YK - CK :: YK + CK : YX$. But $YK - CK = YK - YE + CA - CK = KE - BK$. And $YK + CK = YK - YE + CA + CK = KE + AK$. Wherefore $CY : KE - BK :: KE + AK : YX$. But by *Lemma* 2. $CY : KE + AK :: YX : AK$. Wherefore by Equality $YX : KE - BK :: AK : YX$; or $KE - BK : YX :: YX : AK$. Q. E. D.

Theſe Things being premiſed, the Theorem will be thus demonſtrated.

In the firſt *Lemma*, $YX : AK :: CX : KE$, or $KE \times YX = AK \times CX$; and in the third *Lemma* it was prov'd, that $KE - BK : YX :: YX : AK$. Wherefore, if the Terms of the firſt Ratio of the laſt Proportion be multiply'd by $YX$, it will be $KE \times YX - BK \times YX : XYq :: YX : AK$ that is, $AK \times CX - BK \times YX : YXq :: YX : AK$, and by multiplying the Extremes and Means into themſelves, it will be $AKq \times XC - AK \times BK \times YX = YX$ *cube*. Therefore for $YX$, $AK$, $BK$, and $CX$, re-ſubſtituting $x$, $n$, $\frac{q}{n}$, and $\frac{r}{nn}$, this Æquation will ariſe, *viz.* $r - qx = x^3$. Q. E. D. I need not ſtay to ſhew you the Variations of the Signs, for they will be determin'd according to the different Caſes of the Problem.

Let then an Æquation be propos'd wanting the third Term, as $x^3 + pxx + r = 0$; in order to conſtruct which, take $n$ for any Number of equal Parts; take alſo, in any right Line, two Lengths $KA = \frac{r}{nn}$, and $KB = p$, and let them be taken the ſame Way if $r$ and $p$ have like Signs; but otherwiſe, take them towards contrary Sides. Biſect $BA$ in $C$, and on $K$, as a Center, with the Radius $KC$, deſcribe a Circle, into which accommodate $CX = n$, producing it both Ways. Join $AX$, produce it both Ways. Then, between the Lines $CX$ and $AX$ draw $EY = CA$, ſo that if produc'd it may paſs through the Point $K$; and $KE$ will be the Root of the Æquation. And the Roots will be Affirmative, when the Point $Y$ falls on that Side of $X$ which lies towards $C$; and Negative, when it falls on the contrary

Side of $X$, provided it be $+r$; but if it be $-r$, it will be the Reverſe of this.

To demonſtrate this Propoſition, look back to the Figures and Lemma's of the former; and then you will find it thus.

By *Lemma* 1. $YX : AK :: CX : KE$, or $YX \times KE = AK \times CX$, and by *Lemma* 3, $KE - KB : YX :: YX : AK$, or, (taking $KB$ towards contrary Parts) $KE + KB : YX :: YX : AK$, and therefore $KE + KB$ multiply'd by $KE$ will be to $YX \times KE$ : (or $AK \times CX$) :: $YX : AK$, or as $CX : KE$. Wherefore multiplying the Extreams and Means into themſelves, $KE$ *cube* $+ KB \times KEq = AK \times CXq$; and then for $KE$, $KB$, $AK$, and $CK$, reſtoring their Subſtitutes, you will find the laſt Æquation to be the ſame with what was propos'd, $x^3 + pxx = r$, or $x^3 + pxx \pm r = 0$.

Let an Æquation, having three Dimenſions, and wanting no Term, be propos'd in this Form, $x^3 + pxx + qx + r = 0$, ſome of whoſe Roots ſhall be Affirmative, and ſome Negative.

And firſt ſuppoſe $q$ a Negative Quantity, then in any right Line, as $KB$, let two Lengths be taken, as $KA = \frac{r}{q}$, and $KB = p$, and take them the ſame Way, if $p$ and $\frac{r}{q}$ have contrary Signs; but if their Signs are alike, then take the Lengths contrary Ways from the Point $K$. Biſect $AB$ in $C$, and there erect the Perpendicular $CX$ equal to the Square Root of the Term $q$; then between the Lines $AX$ and $CX$, produc'd infinitely both Ways, inſcribe the right Line $EY = AC$, ſo that being produc'd, it may paſs through $K$; ſo ſhall $KE$ be the Root of the Æquation, which will be Affirmative when the Point $X$ falls between $A$ and $E$; but Negative when the Point $E$ falls on that Side of the Point $X$ which is towards $A$. [*Vide Figure* 95.]

If $q$ had been an Affirmative Quantity, then in the Line $KB$ you muſt have taken thoſe two Lengths thus, *viz.* $KA = \sqrt{\frac{-r}{p}}$, and $KB = \frac{q}{KA}$, and the ſame Way from $K$, if $\sqrt{\frac{-r}{p}}$ and $\frac{q}{KA}$ have different Signs; but contrary Ways, if the Signs are of the ſame Nature. $BA$ alſo muſt be biſected in $C$; and there the Perpendicular $CX$ erected equal to the Term $p$; and between the Lines $AX$ and $CX$, infinitely drawn out both Ways, the right Line $EY$ muſt alſo be inſcribed equal to $AC$, and made to paſs through the Point $K$, as before; then would $XY$ be the Root of the Æquation; Negative when the Point $X$ ſhould fall between $A$ and $E$, and Affirmative when the Point $Y$ falls on the Side of the Point $X$ towards $C$.

## *The Demonſtration of the firſt Caſe.*

By the firſt Lemma, $KE$ was to $CX$ as $AK$ to $YX$, and (by Compoſition) ſo $KE+AK$, i. e. $KY+KC$ is to $CX+YX$, i. e. $CY$. But in the right-angled Triangle $KCY$, $YCq=YKq-KCq=\overline{KY+KC}\times\overline{KY-KC}$; and by reſolving the equal Terms into Proportionals, $KY+KC$ is to $CY$ as $CY$ is to $KY-KC$; or $KE+AK$ is to $CY$ as $CY$ is to $EK-KB$. Wherefore ſince $KE$ was to $XC$ in this Proportion, by Duplication $KEq$ will be to $CXq$ as $KE+AK$ to $KE-KB$, and by multiplying the Extreams and Means by themſelves $KE\,cube-KB\times KEq=CXq\times KE+CXq\times AK$. And by reſtoring the former Values $x^3-pxx=qx+r$.

## *The Demonſtration of the ſecond Caſe.*

By the firſt Lemma, $KE$ is to $CX$ as $AK$ is to $YX$, then by multiplying the Extreams and Means by themſelves, $KE\times YX=CX\times AK$. Therefore in the preceding Caſe, put $KE\times YX$ for $CX\times AK$, and it will be $KE\,cub.-KB\times KEq=CXq\times KE+CX\times KE\times YX$; and by dividing all by $KE$, there will be $KEq-KB\times KE=CXq+CX\times YX$; then multiplying all by $AK$, and you'll have $AK\times KEq-KB\times KA\times KE=AK\times CXq+AK\times CX\times YX$. And again, put $KE\times YX$ inſtead of its equal $CX\times AK$, then $AK\times KEq-AK\times KB\times KE=EK\times CX\times YX+KE\times YXq$; whence all being divided by $KE$ there will ariſe $AK\times KE-AK\times KB=YX\times CX+YXq$; and when all are multiply'd by $YX$ there will be $AK\times KE\times YX-AK\times KB\times YX=YXq\times CX+YX\,cube$. And inſtead of $KE\times YX$ in the firſt Term, put $CX\times AK$, and then $CX\times AKq-AK\times BK\times YX=CX\times$

$YXq + YX\,cube$, or, which is the ſame Thing, $YX\,cube + CX \times YXq + AK \times KB \times YX - CX \times AKq = 0$. And by ſubſtituting for $YX$, $CX$, $AK$, and $KB$, their Values $x$, $p$, $\sqrt{\frac{-r}{p}}$, $q\sqrt[3]{\frac{p}{-r}}$, this Æquation will come out, $x^3 + pxx + qx + r = 0$.

But theſe Æquations are alſo ſolv'd, by drawing a right Line from a given Point, in ſuch a Manner that the Part of it, which is intercepted between another right Line and a Circle, both given in Poſition, may be of a given Length. [*Vide Figure* 96]

For, let there be propos'd a Cubick Æquation $x^3 * + qx + r = 0$, whoſe ſecond Term is wanting. Draw the right Line $KA$ at Pleaſure, which call $n$. In $KA$, produc'd both Ways, take $KB = \frac{q}{n}$ on the ſame Side of the Point $K$ as the Point $A$ is if $q$ be Negative, if not, on the contrary. Biſect $BA$ in $C$, and from the Center $A$, with the Diſtance $AC$, deſcribe a Circle $CX$. To this inſcribe the right Line $CX = \frac{r}{nn}$, and through the Points $K$, $C$, and $X$ deſcribe the Circle $KCXG$. Join $AX$, and produce it till it again cuts the Circle $KCXG$ laſt deſcrib'd in the Point $G$. Laſtly, between this Circle $KCXG$, and the right Line $KC$ produc'd both Ways, inſcribe the right Line $EY = AC$, ſo that $EY$ produc'd paſs through the Point $G$. And $EG$ will be one of the Roots of the Æquation. But thoſe Roots are Affirmative which fall in the greater Segment of the Circle $KGC$, and Negative which fall in the leſſer $KFC$, if $r$ is Negative, and the contrary will be when $r$ is Affirmative.

In order to demonſtrate this Conſtruction, let us premiſe the following *Lemmata*.

LEMMA I. All Things being ſuppos'd as in the Conſtruction, $CE$ is to $KA$ as $CE + CX$ is to $AY$, and as $CX$ to $KY$.

For the right Line $KG$ being drawn, $AC$ is to $AK$ as $CX$ is to $KG$, becauſe the Triangles $ACX$ and $AKG$ are Similar. The Triangles $YEC$, $YKG$ are alſo Similar; for the Angle at $Y$ is common to both Triangles, and the Angles $G$ and $C$ are in the ſame Segment $KCG$ of the Circle $EGCK$, and therefore equal. Whence $CE$ will be to $EY$ as $KG$ to $KY$, that is, $CE$ to $AC$ as $KG$ to $KY$, becauſe $EY$ and $AC$ were ſuppoſed equal. And by comparing this with the Proportionality above, it will follow by Equality of Proportion that $CE$ is to $KA$ as $CX$ to $KY$, and alternately $CE$ is to $CX$ as $KA$ to $KY$. Whence, by Compoſition, $CE+CX$ will be to $CX$ as $KA+KY$ to $KY$, that is, $AY$ to $KY$, and alternately $CE+CX$ is to $AY$ as $CX$ is to $KY$, that is, as $CE$ to $KA$. Q. E. D.

LEMMA II. Let fall the Perpendicular $CH$ upon the right Line $GY$, and the Rectangle $2HEY$ will be equal to the Rectangle $CE\times CX$.

For the Perpendicular $GL$ being let fall upon the Line $AY$, the Triangles $KGL$, $ECH$ have right Angles at $L$ and $H$, and the Angles at $K$ and $E$ are in the ſame Segment $CGK$ of the Circle $CKEG$, and are therefore equal; conſequently the Triangles are Similar. And therefore $KG$ is to $KL$ as $EC$ to $EH$. Moreover, $AM$ being let fall from the Point $A$ perpendicular to the Line $KG$, becauſe $AK$ is equal to $AG$, $KG$ will be biſected in $M$; and the Triangles $KAM$ and $KGL$ are Similar, becauſe the Angle at $K$ is common, and the Angles at $M$ and $L$ are right ones; and therefore $AK$ is to $KM$ as $KG$ is to $KL$. But as $AK$ is to $KM$ ſo is $2AK$ to $2KM$, or $KG$; (and becauſe the Triangles $AKG$ and $ACX$ are Similar) ſo is $2AC$ to $CX$; alſo (becauſe $AC=EY$) ſo is $2EY$ to $CX$. Therefore $2EY$ is to $CX$ as $KG$ to $KL$. But $KG$ was to $KL$ as $EC$ to $EH$, therefore $2EY$ is to $CX$ as $EC$ to $EH$, and ſo the Rectangle $2HEY$ (by multiplying the Extreams and Means by themſelves) is equal to $EC\times CX$. Q. E. D.

Here we took the Lines $AK$ and $AG$ equal. For the Rectangles $CAK$ and $XAG$ are equal (by *Cor.* to 36 *Prop.* of the 3d Book of *Euc.*) and therefore as $CA$ is to $XA$ ſo is $AG$ to $AK$. But $XA$ and $CA$ are equal by Hypotheſis; therefore $AG=AK$.

LEMMA III. All Things being as above, the three Lines $BY$, $CE$, $KA$ are continual Proportionals.

For (by *Prop.* 12. *Book* 2. *Elem.*) $CYq=EYq+CEq+2EY\times EH$. And by taking $EYq$ from both Sides, $CYq-EYq=CEq+2EY\times EH$. But $2EY\times EH=CE\times CX$ (by *Lem.* 2.) and by adding $CEq$ to both Sides, $CEq+$

$2EY\times EH=CEq+CE\times CX$. Therefore $\overline{CYq-EYq}=CEq+CE\times CX$, that is, $\overline{CY+EY}\times\overline{CY-EY}=CEq+CE\times CX$. And by reſolving the equal Rectangles into proportional Sides, it will be as $CE+CX$ is to $CY+EY$, ſo is $CY-EY$ to $CE$. But the three Lines $EY$, $CA$, $CB$, are equal, and thence $CY+EY=CY+CA=AY$, and $CY-EY=CY-CB=BY$. Write $AY$ for $CY+EY$, and $BY$ for $CY-EY$, and it will be as $CE+CX$ is to $YA$ ſo is $BY$ to $CE$. But (by *Lem.* 1.) $CE$ is to $KA$ as $CE+CX$ is to $AY$, therefore $CE$ is to $KA$ as $BY$ is to $CE$, that is, the three Lines $BY$, $CE$, and $KA$ are continual Proportionals. Q. E. D.

Now, by the Help of theſe three Lemmas, we may demonſtrate the Conſtruction of the preceding Problem, thus:

By *Lem.* 1. $CE$ is to $KA$ as $CX$ is to $KY$, ſo $KA\times CX=CE\times KY$, and by dividing both Sides by $CE$, $\frac{KA\times CX}{CE}=KY$. To theſe equal Sides add $BK$, and $BK+\frac{KA\times CX}{CE}=BY$. Whence (by *Lem.* 3.) $BK+\frac{KA\times CX}{CE}$ is to $CE$ as $CE$ is to $KA$, and thence, by multiplying the Extreams and Means by themſelves, $CEq=BK\times KA+\frac{KAq\times CX}{CE}$, and both Sides being multiply'd by $CE$, $CE\ cub.=KB\times KA\times CE+KAq\times CX$. $CE$ was called $x$, the Root of the Æquation $KA=n$, $KB=\frac{q}{n}$, and $CX=\frac{r}{nn}$. Theſe being ſubſtituted inſtead of $CE$, $KA$, $KB$, and $CX$, there will ariſe this Æquation, $x^3=qx+r$, or $x^3-qx-r=0$; when $q$ and $r$ are Negatives, $KA$ and $KB$ having been taken on the ſame Side of the Point $K$, and the Affirmative Root being in the greater Segment $CGK$. This is one Caſe of the Conſtruction to be demonſtrated. Draw $KB$ on the contrary Side, that is, let its Sign be changed, or the Sign of $\frac{q}{n}$, or, which is the ſame Thing, the Sign of the Term $q$, and there will be had the Conſtruction of the Æquation $x^3+qx-r=0$. Which is the other Caſe. In theſe Caſes $CX$, and the Affirmative Root $CE$, fall towards the ſame Parts of the Line $AK$. Let $CX$

and the Negative Root fall towards the ſame Parts when the Sign of $CX$, or $\frac{r}{nn}$, or (which is the ſame Thing) $r$ is changed; and this will be the third Caſe $x^3 + qx + r = 0$, where all the Roots are Negative. And again, when the Sign of $KB$, or $\frac{q}{n}$, or only $q$, is changed, it will be the fourth Caſe $x^3 - qx + r = 0$. The Conſtructions of all theſe Caſes may be eaſily run through, and particularly demonſtrated after the ſame Manner as the firſt was; and with the ſame Words, by changing only the Situation of the Lines.

Now let the Cubick Æquation $x^3 + pxx * + r = 0$, whoſe third Term is wanting, be to be conſtructed.

In the ſame Figure $n$ being taken of any Length, take in any infinite right Line $AY$, $KA$, and $KB = \frac{r}{nn}$, and $p$, and take them on the ſame Side of the Point $K$, if the Signs of the Terms $p$ and $r$ are the ſame, otherwiſe on contrary Sides. Biſect $BA$ in $C$, and from the Center $K$ with the Diſtance $KC$ deſcribe the Circle $CXG$. And to it inſcribe the right Line $CX$ equal to $n$ the aſſumed Length. Join $AX$ and produce it to $G$, ſo that $AG$ may be equal to $AK$, and through the Points $K, C, X, G$ deſcribe a Circle. And, laſtly, between this Circle and the right Line $KC$, produc'd both Ways, draw the right Line $EY = AC$, ſo that being produced it may paſs through the Point $G$; then the right Line $KY$ being produc'd, will be one of the Roots of the Æquation. And thoſe Roots are Affirmative which fall on that Side of the Point $K$ on which the Point $A$ is on, if $r$ is Affirmative; but if $r$ is Negative, then the Affirmative Roots fall on the contrary Side. And if the Affirmative fall on one Side, the Negative fall on the other.

This Conſtruction is demonſtrated by the Help of the three laſt *Lemma's* after this Manner:

By the third *Lemma*, $BY$, $CE$, $KA$ are continual Proportionals; and by *Lemma* 1. as $CE$ is to $KA$ ſo is $CX$ to $KY$. Therefore $BY$ is to $CE$ as $CX$ to $KY$. $BY = KY - KB$. Therefore $KY - KB$ is to $CE$ as $CX$ is to $KY$. But as $KY - KB$ is to $CE$ ſo is $\overline{KY - KB} \times KY$ to $CE \times KY$, by *Prop.* 1. *Book* 6. *Euc.* and becauſe of the Proportionals $CE$ to $KA$ as $CX$ to $KY$ is $CE \times KY = KA \times CX$.

Therefore $\overline{KY-KB}\times KY$ is to $KA\times CX$ (as $KY-KB$ to $CE$, that is, as $CX$ to $KY$. And by multiplying the Extreams and Means by themſelves $\overline{KY-KB}\times KYq = KA\times CXq$; that is, $KY\,cub. - KB\times KYq = KA\times CXq$. But in the Conſtruction $KY$ was $x$ the Root of the Æquation, $KB$ was put $=p$, $KA=\frac{r}{nn}$, and $CX=n$. Write therefore $x$, $p$, $\frac{r}{nn}$, and $n$ for $KY$, $KB$, $KA$, and $CX$ reſpectively, $x^3-pxx$ will be equal to $r$, or $x^3-pxx-r=0$.

This Conſtruction may be reſolv'd into four Caſes of Æquations, $x^3-pxx-r=0$, $x^3-pxx+r=0$, $x^3+pxx-r=0$, and $x^3+pxx+r=0$. The firſt Caſe I have already demonſtrated; the reſt are demonſtrated with the ſame Words, only changing the Situation of the Lines. To wit, as in taking $KA$ and $KB$ on the ſame Side of the Point $K$, and the Affirmative Root $KY$ on the contrary Side, has already produc'd $KY\,cub. - KB\times KYq = KA\times CXq$, and thence $x^3-pxx-r=0$; ſo by taking $KB$ on the other Side the Point $K$, it will produce, by the like Reaſoning, $KY\,cub. + KYq\times KB = KA\times CXq$, and thence $x^3+pxx-r=0$. And in theſe two Caſes, if the Situation of the Affirmative Root $KY$ be changed, by taking it on the other Side of the Point $K$, by a like Series of Arguments, it will fall into the other two Caſes, $KY\,cub. + KB\times KYq = -KA\times CXq$, or $x^3+pxx+r=0$, and $KY\,cub. - KB\times KYq = -KA\times CXq$, or $x^3-pxx+r=0$. Which were all the Caſes to be demonſtrated.

Now let this Cubick Æquation $x^3+pxx+qx+r=0$ be propos'd, wanting no Term (unleſs perhaps the third). Which is conſtructed after this Manner: [*Vide Figures 97 and 98.*]

Take $n$ at Pleaſure. Draw any right Line $GC=\frac{n}{2}$, and at the Point $G$ erect a Perpendicular $GD=\sqrt{\frac{r}{p}}$, and if the Terms $p$ and $r$ have contrary Signs, from the Center $C$, with the Interval $CD$ deſcribe a Circle $PBE$. If they have the ſame Signs, from the Center $D$, with the Space $GC$, deſcribe an occult Circle, cutting the right Line $GA$ in

H;

*H*; then from the Center *C*, with the Diſtance *GH*, deſcribe the Circle *PBE*. Then make $GA = -\frac{q}{n} - \frac{r}{np}$ on the ſame Side the Point *G* that *C* is on, if now the Quantity $-\frac{q}{n} - \frac{r}{np}$ (the Signs of the Terms *p*, *q*, *r* in the Æquation to be conſtructed being well obſerv'd) ſhould come out Affirmative; otherwiſe, draw *GA* on the other Side of the Point *G*, and at the Point *A* erect the Perpendicular *AY*, between which and the Circle *PBE* already deſcrib'd, draw the right Line *EY* equal to *p*, ſo that being produc'd, it may paſs through the Point *G*; which being done, the Line *EG* will be one of the Roots of the Æquation to be conſtructed. Thoſe Roots are Affirmative where the Point *E* falls between the Points *G* and *Y*, and Negative, where the Point *E* falls without, if *p* is Affirmative; and the contrary, if Negative.

In order to demonſtrate this Conſtruction, let us premiſe the following Lemmas.

Lemma I. Let *EF* be let fall perpendicular to *AG*, and the right Line *EC* be drawn; $EGq + GCq = ECq + 2CGF$. For (by *Prop.* 12. *Book* 2. *Elem.*) $EGq = ECq + GCq + 2GCF$. Let $GCq$ be added on both Sides, and $EGq + GCq = ECq + 2GCq + 2GCF$. But $2GCq + 2GCF = 2GC \times \overline{GC + CF} = 2CGF$. Therefore $EGq + GCq = ECq + 2CGF$. Q.E.D.

Lemma II. In the firſt Caſe of the Conſtruction, where the Circle *PBE* paſſes through the Point *D*, $GEq - GDq = 2CGF$. For by the firſt Lemma $EGq + GCq = ECq + 2CGF$, and by taking $CGq$ from both Sides, $EGq = ECq - GCq + 2CGF$. But $ECq - GCq = CDq - GCq = GDq$. Therefore $EGq = GDq + 2CGF$, and by taking $GDq$ from both Sides, $EGq - GDq = 2CGF$. Q.E.D.

Lemma III. In the ſecond Caſe of the Conſtruction, where the Circle *PCD* does not paſs through the Point *D*, $EGq + GDq = 2CGF$. For, by the firſt Lemma, $EGq + GCq = ECq + 2CGF$. Take $ECq$ from both Sides, and $EGq + GCq - ECq = 2CGF$. But $GC = DH$, and $EC$

$= CP = GH$. Therefore $GCq - ECq = DHq - GHq = GDq$, and ſo $EGq + GDq = 2CGF$. Q. E. D.

LEMMA IV. $GY \times 2CGF = 2CG \times AGE$. For, by reaſon of the ſimilar Triangles $GEF$ and $GYA$, as $GF$ is to $GE$ ſo is $AG$ to $GY$, that is, (by *Prop.* 1. *Book* 6. *Elem.*) as $2CG \times AG$ is to $2CG \times GY$. Let the Extreams and Means be multiply'd by themſelves, and $2CG \times GY \times GF = 2CG \times AG \times GE$. Q. E. D.

Now, by the Help of theſe Lemmas, the Conſtruction of the Problem may be thus demonſtrated.

In the firſt Caſe, $EGq - GDq = 2CGF$ (by Lemma 2.) and by multiplying all by $GY$, $EGq \times GY - GDq \times GY = 2CGF \times GY =$ (by Lemma 4.) $2CG \times AGE$. Inſtead of $GY$ write $EG + EY$, and $EG\,cub. + EY \times EGq - GDq \times EG - GDq \times EY = 2CGA \times EG$, or $EG\,cub. + EY \times \overline{EGq - GDq - 2CGA} \times EG - GDq \times EY = 0$.

In the ſecond Caſe, $EGq + GDq = 2CGF$ (by Lemma 3.) and by multiplying all by $GY$, $EGq \times GY + GDq \times GY = 2CGF \times GY = 2CG \times AGE$, by Lemma 4. Inſtead of $GY$ write $EG + EY$, and $EG\,cub. + EY \times EGq + GDq + EG + GDq \times EY = 2CGA \times EG$, or $EG\,cub. + EY \times EGq + \overline{GDq - 2CGA} \times EG + GDq \times EY = 0$.

But the Root of the Æquation $EG = x$, $GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2CG = n$, and $GA = -\frac{q}{n} - \frac{r}{np}$, that is, in the firſt Caſe, where the Signs of the Terms $p$ and $r$ are different; but in the ſecond Caſe, where the Sign of one of the two, $p$ or $r$, is changed, there is $-\frac{q}{n} + \frac{r}{np} = GA$. Let therefore $EG$ be put $= x$, $GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2CG = n$, and $GA = -\frac{q}{n} \mp \frac{r}{np}$, and in the firſt Caſe it will be $x^3 + px^2 + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x - r = 0$; that is, $x^3 + px^2 + qx - r = 0$; but in the ſecond Caſe, $x^3 + pxx + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x + r = 0$, that is, $x^3 + px^2 + qx + r = 0$.

$= 0$. Therefore in both Caſes $EG$ is the true Value of the Root $x$. Q. E. D.

But either Caſe may be diſtinguiſh'd into its ſeveral Particulars; as the former into theſe, $x^3 + px^2 + qx - r = 0$, $x^3 + px^2 - qx - r = 0$, $x^3 - px^2 + qx + r = 0$, $x^3 - px^2 - qx + r = 0$, $x^3 + px^2 - r = 0$, and $x^3 - px^2 + r = 0$; the latter into theſe, $x^3 + px^2 + qx + r = 0$, $x^3 + px^2 - qx + r = 0$, $x^3 - px^2 + qx - r = 0$, $x^3 - px^2 - qx - r = 0$, $x^3 + px^2 + r = 0$, and $x^3 - px^2 - r = 0$. The Demonſtration of all which Caſes may be carry'd on in the ſame Words with the two already demonſtrated, by only changing the Situation of the Lines.

Theſe are the chief Conſtructions of Problems, by inſcribing a right Line given in Length ſo between a Circle and a right Line given in Poſition, that the inſcrib'd right Line produc'd may paſs through a given Point. And ſuch a right Line may be inſcrib'd by deſcribing a Conchoid, of which let that Point, through which the right Line given ought to paſs, be the Pole, the other right Line given in Poſition; the Ruler or Aſymptote, and the Interval, the Length of the inſcrib'd Line. For this Conchoid will cut the Circle in the Point $E$, through which the right Line to be inſcrib'd muſt be drawn. But it will be ſufficient in Practice to draw the right Line between a Circle and a right Line given in Poſition by any Mechanick Method.

But in theſe Conſtructions obſerve, that the Quantity $n$ is undetermin'd and left to be taken at Pleaſure, that the Conſtruction may be more conveniently fitted to particular Problems. We ſhall give Examples of this in finding two mean Proportionals, and in triſecting an Angle.

Let $x$ and $y$ be two mean Proportionals to be found between $a$ and $b$. Becauſe $a, x, y, b$ are continual Proportionals, $a^2$ will be to $x^2$ as $x$ to $b$, therefore $x^3 = baa$, or $x^3 - aab = 0$. Here the Terms $p$ and $q$ of the Æquation are wanting, and $-aab$ is in the room of the Term $r$; therefore in the firſt Form of the Conſtructions, where the right Line $EY$ tending to the given Point $K$, is drawn between other two right Lines, $EX$ and $YC$, given in Poſition, and ſuppoſe the right Line $CX = \frac{r}{nn} = \frac{-aab}{nn}$, let $n$ be taken equal to $a$, and then $CX$ will be $= -b$. From whence the like Conſtruction comes out. [*Vide Figure* 99.]

I draw any Line, $KA = a$, and biſect it in $C$, and from the Center , with the Diſtance $KC$, deſcribe the Circle $CX$, to which I inſcribe the right Line $CX = b$, and between $AX$ and $CX$. infinitely produc'd, I ſo inſcribe $EY = CA$, that $EY$ being produc'd, may paſs through the Point $K$. So $KA$, $XY$; $KE$, $CX$ will be continual Proportionals, that is, $XY$ and $KE$ two mean Proportionals between $a$ and $b$. This Conſtruction is known. [*Vide Figure* 100.]

But in the other Form of the Conſtructions, where the right Line $EY$ converging to the given Point $G$ is inſcrib'd between the Circle $GECX$ and the right Line $AK$, and $CX = \frac{r}{nn}$, that is, (in this Problem) $= \frac{-aab}{nn}$, I put, as before, $n = a$, and then $CX$ will be $= b$, and the reſt are done as follows. [*Vide Figure* 101.]

I draw any right Line $KA = a$, and biſect it in $C$, and from the Center $A$, with the Diſtance $AK$, I deſcribe the Circle $KG$, to which I inſcribe the right Line $KG = 2b$, conſtituting the *Iſoſceles* Triangle $AKG$. Then, through the Points $C$, $K$, $G$ I deſcribe the Circle, between the Circumference of which and the right Line $AK$ produc'd, I inſcribe the right Line $EY = CK$ tending to the Point $G$. Which being done, $AK$, $EC$, $KY$, $\frac{1}{2}KG$ are continual Proportionals, that is, $EC$ and $KY$ are two mean Proportionals between the given Quantities $a$ and $b$.

Let there be an Angle to be divided into three equal Parts; [*Vide Figure* 102.] and let that Angle be $ACB$, and the Parts thereof to be found be $ACD$, $ECD$, and $ECB$; from the Center $C$, with the Diſtance $CA$, let the Circle $ADEB$ be deſcrib'd, cutting the right Lines $CA$, $CD$, $CE$, $CB$ in $A$, $D$, $E$, $B$. Let $AD$, $DE$, $EB$ be join'd, and $AB$ cutting the right Lines $CD$, $CE$ at $F$ and $H$, and let $DG$, meeting $AB$ in $G$, be drawn parallel to $CE$. Becauſe the Triangles $CAD$, $ADF$, and $DFG$ are Similar, $CA$, $AD$, $DF$, and $FG$ are continual Proportionals. Therefore if $AC = a$, and $AD = x$, $DF$ will be equal to $\frac{xx}{a}$, and $FG = \frac{x^3}{aa}$. And $AB = BH + HG + FA - GF = 3AD - GF = 3x - \frac{x^3}{aa}$. Let $AB = b$, then $b = 3x - \frac{x^3}{aa}$, or $x^3 - 3aax + aab = 0$. Here $p$, the ſecond Term of the Æquation, is wanting, and inſtead of $q$ and $r$ we have $-3aa$ and $aab$. Therefore in the firſt Form of the Conſtructions, where $p$ was $=0$, $KA=n$, $KB=\frac{q}{n}$, and $CX=\frac{r}{nn}$, that is, in this Problem, $KB=-\frac{3aa}{n}$, and $CX=\frac{aab}{nn}$, that theſe Quantities may come out as ſimple as poſſible, I put $n=a$, and ſo $KB=-3a$, and $CX=b$. Whence this Conſtruction of the Problem comes out.

Draw any Line, $KA=a$, and on the contrary Side make $KB=3a$. [*Vide Figure* 103.] Biſect $BA$ in $C$, and from the Center $K$, with the Diſtance $KC$, deſcribe a Circle, to which inſcribe the right Line $CX=b$, and the right Line $AX$ being drawn between that infinitely produc'd and the right Line $CX$, inſcribe the right Line $EY=AC$, and ſo that it being produc'd, will paſs through the Point $K$. So $XY$ will be $=x$. But (ſee the laſt Figure) becauſe the Circle $ADEB=$ Circle $CXA$, and the Subtenſe $AB=$ Subtenſe $CX$, and the Parts of the Subtenſes $BH$ and $XY$ are equal; the Angles $ACB$, and $CKX$ will be equal, as alſo $BCH$, $XKY$; and ſo the Angle $XKY$ will be one third Part of the Angle $CKX$. Therefore the third Part $XKY$ of any given Angle $CKX$ is found by inſcribing the right Line $EY=AC$, the Diameter of the Circle between the Chords $CX$ and $AX$ infinitely produc'd, and converging at $K$ the Center of the Circle.

Hence, if from $K$, the Center of the Circle, you let fall the Perpendicular $KH$ upon the Chord $CX$, the Angle $HKY$ will be one third Part of the Angle $HKX$; ſo that if any Angle $HKX$ were given, the third Part thereof $HKY$ may be found by letting fall from any Point $X$ of any Side $KX$, the Line $HX$ perpendicular to the other Side $HK$, and by drawing $XE$ parallel to $HK$, and by inſcribing the right Line $YE=2XK$ between $XH$ and $XE$, ſo that it being produc'd may paſs through the Point $K$. Or thus. [*Vide Figure* 104.]

Let any Angle $AXK$ be given. To one of its Sides $AX$ raiſe a Perpendicular $XH$, and from any Point $K$ of the other Side $XK$ let there be drawn the Line $KE$, the Part of which $EY$ (lying between the Side $AX$ produc'd, and the Perpendicular $XH$) is double the Side $XK$, and the Angle $KEA$ will be one third of the given Angle $AXK$. Again,

Again; the Perpendicular *EZ* being raiſ'd, and *KF* being drawn, whoſe Part *ZF*, between *EF* and *EZ*, let be double to *KE*, and the Angle *KFA* will be one third of the Angle *KEA*; and ſo you may go on by a continual Triſection of an Angle *ad infinitum*. This Method is in the 32d *Prop.* of the 4th Book of *Pappus*.

If you would triſect an Angle by the other Form of Conſtructions, where the right Line is to be inſcrib'd between another right Line and a Circle, here alſo will $KB = \frac{q}{n}$, and $CX = \frac{r}{nn}$, that is, in the Problem we are now about, $KB = \frac{-3aa}{n}$, and $CX = \frac{aab}{nn}$; and ſo by putting $n = a$, $KB$ will be $= -3a$, and $CX = b$. Whence this Conſtruction comes out.

From any Point *K* let there be drawn two right Lines towards the ſame Way, $KA = a$, and $KB = 3a$. [*Vide Figure* 105.] Biſect *AB* in *C*, and from the Center *A* with the Diſtance *AC* deſcribe a Circle. To which inſcribe the right Line $CX = b$. Join *AX*, and produce it till it cuts the Circle again in *G*. Then between this Circle and the right Line *AC*, infinitely produc'd, inſcribe the Line $EY = AC$, and paſſing through the Point *G*; and the right Line *EC* being drawn, will be equal to *x* the Quantity ſought, by which the third Part of the given Angle will be ſubtended.

This Conſtruction ariſes from the Form above; which, however, comes out better thus: Becauſe the Circles *ADEB* and *KXG* are equal, and alſo the Subtenſes *CX* and *AB*, the Angles *CAX* or *KAG*, and *ACB* are equal, therefore *CE* is the Subtenſe of one third Part of the Angle *KAG*. Whence in any given Angle *KAG*, that its third Part *CAE* may be found, inſcribe the right Line *EY* equal to the Semi-Diameter *AG* of the Circle *KCG*, between the Circle and the Side *KA*, of the Angle, infinitely produc'd, and tending to the Point *G*. Thus *Archimedes*, in *Lemma* 8. taught to triſect an Angle. The ſame Conſtructions may be more eaſily explain'd than I have done here; but in theſe I would ſhow how, from the general Conſtructions of Problems I have already explain'd, we may derive the moſt ſimple Conſtructions of particular Problems.

Beſides the Conſtructions here ſet down, we might add many more. [*Vide Figure* 106.] As if there were two mean Proportionals to be found between $a$ and $b$. Draw any right Line $AK = b$, and perpendicular to it $AB = a$. Biſect $AK$ in $I$, and in $AK$ put $AH$ equal to the Subtenſe $BI$; and alſo in the Line $AB$ produc'd, $AC =$ Subtenſe $BH$. Then in the Line $AK$ on the other Side of the Point $A$ take $AD$ of any Length and $DE$ equal to it, and from the Centers $D$ and $E$, with the Diſtances $DB$ and $EC$, deſcribe two Circles, $BF$ and $CG$, and between them draw the right Line $FG$ equal to the right Line $AI$, and converging at the Point $A$, and $AF$ will be the firſt of the two mean Proportionals that were to be found.

The Ancients taught how to find two mean Proportionals by the Ciſſoid; but no Body that I know of hath given a good manual Deſcription of this Curve. [*Vide Figure* 107.] Let $AG$ be the Diameter, and $F$ the Center of a Circle to which the Ciſſoid belongs. At the Point $F$ let the Perpendicular $FD$ be erected, and produc'd *in infinitum*. And let $FG$ be produc'd to $P$, that $FP$ may be equal to the Diameter of the Circle. Let the Ruler $PED$ be moved, ſo that the Leg $EP$ may always paſs through the Point $P$, and the other Leg $ED$ muſt be equal to the Diameter $AG$, or $FP$, with its End $D$, always moving in the Line $FD$; and the middle Point $C$ of this Leg will deſcribe the Ciſſoid $GCK$ which was deſired, as has been already ſhewn. Wherefore, if between any two Quantities, $a$ and $b$, there be two mean Proportionals to be found: Take $AM = a$, raiſe the Perpendicular $MN = b$. Join $AN$, and move the Ruler $PED$, as was juſt now ſhewn, until its Point $C$ fall upon the right Line $AN$. Then let fall $CB$ perpendicular to $AP$, take $t$ to $BH$, and $v$ to $BG$, as $MN$ is to $BC$, and becauſe $AB$, $BH$, $BG$, $BC$ are continual Proportionals, $a$, $t$, $v$, $b$ will alſo be continual Proportionals.

By the Application of ſuch a Ruler other ſolid Problems may be conſtructed.

Let there be propoſed the Cubick Æquation $x^3\ pxx - qx + r = 0$; where $q$ is always Negative, $r$ Affirmative, and $p$ of any Sign. Make $AG = \frac{r}{q}$, and biſect it in $F$, and take $FR = \frac{p}{2}$, and that towards $A$ if $p$ is Affirmative, if not towards $P$. Moreover, make $AB = \sqrt{q}$, and erect the

Perpendiculars $FD$ and $BC$. And in the Leg $ED$ of the Ruler, take $ED = AG$ and $EC = AR$; then let the Leg of the Ruler be apply'd to the Scheme, ſo that the Point $D$ may touch the Line $FD$, and the Point $C$ the right Line $BC$, and $BC$ will be the Root of the Æquation ſought, $= x$.

Thus far, I think, I have expounded the Conſtruction of ſolid Problems by Operations whoſe manual Practice is moſt ſimple and expeditious. So the Antients, after they had obtain'd a Method of ſolving theſe Problems by a Compoſition of ſolid Places, thinking the Conſtructions by the Conick Sections uſeleſs, by reaſon of the Difficulty of deſcribing them, ſought eaſier Conſtructions by the Conchoid, Ciſſoid, the Extenſion of Threads, and by any Mechanick Application of Figures. Since uſeful Things, though Mechanical, are juſtly preferable to uſeleſs Speculations in Geometry, as we learn from *Pappus*. So the great *Archimedes* himſelf neglected the Triſection of an Angle by the Conick Sections, which had been handled by other Geometricians before him, and taught how to triſect an Angle in his Lemma's as we have already explain'd. If the Antients had rather conſtruct Problems by Figures not receiv'd in Geometry in that Time, how much more ought theſe Figures now to be preferr'd which are receiv'd by many into Geometry as well as the Conick Sections.

However, I don't agree to this new Sort of Geometricians, who receive all Figures into Geometry. Their Rule of admitting all Lines to the Conſtruction of Problems in that Order in which the Æquations, whereby the Lines are defin'd, aſcend to the Number of Dimenſions, is arbitrary and has no Foundation in Geometry. Nay, it is falſe; for according to this Rule, the Circle ſhould be joined with the Conick Sections, but all Geometers join it with the right Line; and this being an inconſtant Rule, takes away the Foundation of admitting into Geometry all Analytick Lines in a certain Order. In my Judgment, no Lines ought to be admitted into plain Geometry beſides the right Line and the Circle. Unleſs ſome Diſtinction of Lines might be firſt invented, by which a circular Line might be joined with a right Line, and ſeparated from all the reſt. But truly plain Geometry is not to be augmented by the Number of Lines. For all Figures are plain that are admitted into plain Geometry, that is, thoſe which the Geometers poſtulate to be deſcribed *in plano*. And every plain Problem is that which may be conſtructed by plain Figures. So therefore admitting the Conick Sections and other Figures more compounded into plain Geometry, all the ſolid and more than ſolid Problems that can be conſtructed by theſe Figures will become plane. But all plane Problems are of the ſame Order. A right Line Analytically is more ſimple than a Circle; nevertheleſs, Problems which are conſtructed by right Lines alone, and thoſe that are conſtructed by Circles, are of the ſame Order. Theſe Things being poſtulated, a Circle is reduc'd to the ſame Order with a right Line. And mnch more the Ellipſe, which differs much leſs from a Circle than a Circle from a right Line, by poſtulating the right Deſcription thereof *in plano*, will be reduc'd to the ſame Order with the Circle. If any, in conſidering the Ellipſe, ſhould fall upon ſome ſolid Problem, and ſhould conſtruct it by the Help of the ſame Ellipſe, and a Circle: This would be counted a plane Problem, becauſe the Ellipſe was ſuppos'd to be deſcrib'd *in plano*, and every Conſtruction beſides will be ſolv'd by the Deſcription of the Circle only. Wherefore, for the ſame Reaſon, every plane Problem whatever may be conſtructed by a given Ellipſe.

For Example, [*Vide Figure* 108.] If the Center O of the given Ellipſe $ADFG$ be requir'd, I would draw the Parallels $AB$, $CD$ meeting the Ellipſe in $A$, $B$, $C$, $D$; and alſo two other Parallels $EF$, $GH$ meeting the Ellipſe in $E$, $F$, $G$, $H$, and I would biſect them in $I$, $K$, $L$, $M$, and produce $IK$, $LM$, till they meet in $O$. This is a real Conſtruction of a plane Problem by an Ellipſe. There is no Reaſon that an Ellipſe muſt be Analytically defin'd by an Æquation of two Dimenſions. Nor that it ſhould be generated Geometrically by the Section of a ſolid Figure. The Hypotheſis, only conſidering it as already deſcrib'd *in plano*, reduces all ſolid Problems conſtructed by it to the Order of plane ones, and concludes, that all plane ones may be rightly conſtructed by it. And this is the State of the *Poſtulate*. But perhaps, by the Power of Poſtulates it is lawful to mix that which is now done, and that which is given. Therefore let this be a Poſtulate to deſcribe an Ellipſe *in plano*, and then all thoſe Problems that can be conſtructed by an Ellipſe, may be reduc'd to the Order of plane ones, and all plane Problems may be conſtructed by the Ellipſe.

It is neceſſary therefore that either plane and ſolid Problems be confuſed among one another, or that all Lines be flung out of plane Geometry, beſides the right Line and the

Circle, unleſs it happens that ſometime ſome other is given in the State of conſtructing ſome Problem. But certainly none will permit the Orders of Problems to be confuſed. Therefore the Conick Sections and all other Figures muſt be caſt out of plane Geometry, except the right Line and the Circle, and thoſe which happen to be given in the State of the Problems. Therefore all theſe Deſcriptions of the Conicks *in plano*, which the Moderns are ſo fond of, are foreign to Geometry. Nevertheleſs, the Conick Sections ought not to be flung out of Geometry. They indeed are not deſcribed Geometrically *in plano*, but are generated in the plane Superficies of a geometrical Solid. A Cone is conſtituted geometrically, and cut by a Geometrical Plane. Such a Segment of a Cone is a Geometrical Figure, and has the ſame Place in ſolid Geometry, as the Segment of a Circle has in Plane, and for this Reaſon its Baſe, which they call a Conick Section, is a Geometrical Figure. Therefore a Conick Section hath a Place in Geometry ſo far as the Superficies is of a Geometrical Solid; but is Geometrical for no other Reaſon than that it is generated by the Section of a Solid, and therefore was not in former Times admitted only into ſolid Geometry. But ſuch a Generation is difficult, and generally uſeleſs in Practice, to which Geometry ought to be moſt ſerviceable. Therefore the Antients betook themſelves to various Mechanical Deſcriptions of Figures *in plano*. And we, after their Example, have handled in the preceding Conſtructions. Let theſe Conſtructions be Mechanical; and ſo the Conſtructions by Conick Sections deſcrib'd *in plano* be Mechanical. Let the Conſtructions by Conick Sections given be Geometrical; and ſo the Conſtructions by any other given Figures are Geometrical, and of the ſame Order with the Conſtructions of plane Problems. There is no Reaſon that the Conick Sections ſhould be preferr'd in Geometry before any other Figures, unleſs ſo far as they are deriv'd from the Section of a Cone; they being generally unſerviceable in Practice in the Solution of Problems. But leaſt I ſhould altogether neglect Conſtructions by the Conick Sections, it will be proper to ſay ſomething concerning them, in which alſo we will conſider ſome commodious manual Deſcription.

The Ellipſe is the moſt ſimple of the Conick Sections, moſt known, and neareſt of Kin to a Circle, and eaſieſt deſcrib'd by the Hand *in plano*. Though many prefer the

Parabola before it, for the Simplicity of the Æquation by which it is expreſs'd. But by this Reaſon the Parabola ought to be preferr'd before the Circle it ſelf, which it never is. Therefore the reaſoning from the Simplicity of the Æquation will not hold. The modern Geometers are too fond of the Speculation of Æquations. The Simplicity of theſe is of an Analytick Conſideration. We treat of Compoſition, and Laws are not given to Compoſition from Analyſis; Analyſis does lead to Compoſition: But it is not true Compoſition before its freed from Analyſis. If there be never ſo little Analyſis in Compoſition, that Compoſition is not yet true. Compoſition in it ſelf is perfect, and far from a Mixture of Analytick Speculations. The Simplicity of Figures depend upon the Simplicity of their Geneſis and Ideas, and an Æquation is nothing elſe than a Deſcription (either Geometrical or Mechanical) by which a Figure is generated and rendered more eaſy to the Conception. Therefore we give the Ellipſe the firſt Place, and ſhall now ſhow how to conſtruct Æquations by it.

Let there be any Cubick Æquation propos'd, $x^3 = px^2 + qx + r$, where $p$, $q$, and $r$ ſignify given Co-efficients of the Terms of the Æquations, with their Signs $+$ and $-$, and either of the Terms $p$ and $q$, or both of them, may be wanting. For ſo we ſhall exhibit the Conſtructions of all Cubick Æquations in one Operation, which follows:

From the Point $B$ in any given right Line, take any two right Lines, $BC$ and $BE$, on the ſame Side the Point $B$; and alſo $BD$, ſo that it may be a mean Proportional between them. [*Vide Figure* 109] And call $BC$, $n$, in the ſame right Line alſo take $BA = \frac{q}{n}$, and that towards the Point $C$, if $-q$, if not, the contrary Way. At the Point $A$ erect a Perpendicular, and in it take $AF = p$, $FG = AF$, $FI = \frac{r}{nn}$, and $FH$ to $FI$ as $BC$ is to $BE$. But $FH$ and $FI$ are to be taken on the ſame Side of the Point $F$ towards $G$, if the Terms $p$ and $r$ have the ſame Signs; and if they have not the ſame Signs, towards the Point $A$. Let the Parallelograms $IACK$ and $HAEL$ be compleated, and from the Center $K$, with the Diſtance $KG$, let a Circle be deſcrib'd. Then in the Line $HL$ let there be taken $HR$ on either Side the Point $H$, which let be to $HL$ as

*BD* to *BE*; let *GR* be drawn, cutting *EL* in *S*, and let the Line *GRS* be moved with its Point *R* falling on the Line *HL*, and the Point *S* upon the Line *EL*, until the Point *G* in deſcribing the Ellipſe, meet the Circle, as is to be ſeen in the Poſition of $\gamma\varrho\sigma$. For half the Perpendicular $\gamma X$ let fall, from $\gamma$ the Point of meeting, to *AE* will be the Root of the Æquation. But *G* or $\gamma$ is the End of the Rule *GRS*, or $\gamma\varrho\sigma$, meeting the Circle in as many Points as there are poſſible Roots. And thoſe Roots are Affirmative which fall towards the ſame Parts of the Line *EA*, as the Line *FI* drawn from the Point *F* does, and thoſe are Negative which fall towards the contrary Parts of the Line *AE* if *r* is Affirmative; and contrarily if *r* is Negative.

But this Conſtruction is demonſtrated by the Help of the following Lemma's.

Lemma I. All being ſuppos'd as in the Conſtruction, $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + 2AG \times FI$.

For from the Nature of the Circle, $K\gamma q - CXq = \overline{\gamma X - AI}|^2$. But $K\gamma q = GIq + ACq$, and $CXq = \overline{AX - AC}|^2$, that is, $= AXq - 2CAX + ACq$, and ſo their Difference $GIq + 2CAX - AXq = \overline{\gamma X - AI}|^2 = \gamma Xq - 2AI \times \gamma X + AIq$. Subtract $G1q$ from both, and there will remain $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + AIq - GIq$. But (by *Prop.* 4. *Book* 2. *Elem.*) $AIq = AGq + 2AGI + GIq$, and ſo $AIq - GIq = AGq + 2AGI$, that is, $= 2AG \times \frac{1}{2}\overline{AG + GI}$, or $= 2AG \times FI$, and thence $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + 2AG \times FI$. Q. E. D.

Lemma II. All Things being conſtructed as above $2EAX - AXq = \frac{FI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AG \times FI$.

For it is known, that the Point $\gamma$, by the Motion of the Ruler $\gamma\varrho\sigma$ aſſign'd above, deſcribes an Ellipſe, the Center whereof is *L*, and the two Axis coincide with the two right Lines *LE* and *LH*, of which that which is in $LE = 2\gamma\varrho$, or $= 2GR$, and the other which is in $LH = 2\gamma\sigma$, or $= 2GS$. And the Ratio of theſe to one another is the ſame as that of the Line *HR* to the Line *HL*, or of the Line *BD* to the Line *BE*. Therefore the *Latus Tranſverſum*

ſum is to the principal *Latus Rectum*, as $BE$ is to $BC$, or as $FI$ is to $FH$. Wherefore ſince $\gamma T$ is rightly apply'd to $HL$, it will be from the Nature of the Ellipſe that $GSq - LTq = \frac{FI}{FH} T\gamma$ ſquar'd. But $LT = AE - AX$, and $T\gamma = X\gamma - AH$, let the Squares of which be put inſtead of $LTq$ and $T\gamma q$, and then $GSq - AEq + 2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq$. But $GSq - AEq = \overline{GH + LS}|^2$, becauſe $GS$ is the Hypothenuſe of a Rectangled Triangle, the Sides whereof are equal to $AE$ and $GH + LS$. And (by reaſon of the ſimilar Triangles $RGH$ and $RSL$) $LS$ is to $GH$ as $LR$ is to $HR$, and by Compoſition $GH + LS$ is to $GH$ as $HL$ is to $HR$, and by ſquaring the Proportions $\overline{GH + LS}|^2$ is to $GHq$ as $HLq$ is to $HRq$, that is, (by Conſtruction) as $BEq$ is to $BDq$, that is, as $BE$ is to $BC$, or as $FI$ is to $FH$, and ſo $\overline{GH + LS}|^2 = \frac{FI}{FH} GHq$; i. e. $GSq - AEq = \frac{FH}{FI} GHq$, and ſo $\frac{FI}{FH} GHq + 2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq$, and by ſubtracting $\frac{FI}{FH} GHq$ from both Sides, there will remain $2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq - GHq$. But $AH = AG + GH$, and ſo $AHq = AGq + 2AGH + GHq$, and by ſubtracting $GHq$ from both, there will remain $AHq - GHq = AGq + 2AGH$, that is, $= 2AG \times \overline{\tfrac{1}{2} AG + GH} = 2AG \times FH$, and therefore $2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + 2AG \times FH$, that is, $= \frac{FI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AG \times FI$. Q. E. D.

LEMMA III. All Things ſtanding as before, $AX$ will be to $X\gamma - AG$ as $X\gamma$ is to $2BC$.

For if from the Equals in the ſecond Lemma there be ſubtracted the Equals in the firſt Lemma, there will remain

$2CE \times AX = \frac{HI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AI \times X\gamma$. Let both Sides be multiply'd by $FH$, and $2FH \times CE \times AX = HI \times X\gamma q - 2FI \times AH \times X\gamma + 2AI \times FH \times X\gamma$. But $AI = HI + AH$, and ſo $2FI \times AH - 2FH \times AI = 2FI \times AH - 2FHA - 2FHI$. But $2FI \times HA - 2FHA = 2AHI$, and $2AHI - 2FHI = 2HI \times AF$. Therefore $2FI \times AH - 2FH \times AI = 2HI \times AF$, and ſo $2FH \times CE \times AX = HI \times X\gamma q - 2HI \times AF \times X\gamma$. And thence as $HI$ is to $FH$, ſo is $2CE \times AX$ to $X\gamma q - 2AF \times X\gamma$. But by Conſtruction $HI$ is to $FH$ as $CE$ is to $BC$, and ſo as $2CE \times AX$ is to $2BC \times AX$, and thence $2BC \times AX = X\gamma q - 2AF \times X\gamma$, (by *Prop.* 9. *Book* 5. *Elem.*) But becauſe the Rectangles are equal, the Sides are proportional, $AX$ to $X\gamma - 2AF$, (that is, $X\gamma - AG$) as $X\gamma$ is to $2BC$. Q. E. D.

LEMMA IV. The ſame Things being ſtill ſuppos'd, $2FI$ is to $AX - 2AB$ as $X\gamma$ is to $2BC$.

For if from the Equals in the third Lemma, to wit, $2BC \times AX = X\gamma q - 2AF \times X\gamma$, the Equals in the firſt Lemma be ſubtracted, there will remain $-2AB \times AX + AXq = 2FI \times X\gamma - 2AG \times FI$, that is, $AX \times \overline{AX - 2AB} = 2FI \times \overline{X\gamma - AG}$. But becauſe the Rectangles are Equal, the Sides are Proportional, $2FI$ is to $AX - 2AB$ as $AX$ is to $X\gamma - AG$, that is, (by the third Lemma) as $X\gamma$ is to $2BC$. Q. E. D.

At length, by the Help of theſe Lemma's, the Conſtruction of the Problem is thus demonſtrated.

By the fourth Lemma, $X\gamma$ is to $2BC$ as $2FI$ is to $AX - 2AB$; that is, (by *Prop.* 1. *Book* 6. *Elem.*) as $2BC \times 2FI$ is to $2BC \times \overline{AX - 2AB}$, or to $2BC \times AX - 2BC \times 2AB$. But by the third Lemma, $AX$ is to $X\gamma - 2AF$ as $X\gamma$ is to $2BC$, or $2BC \times AX = X\gamma q - 2AF \times X\gamma$, and ſo $X\gamma$ is to $2BC$ as $2BC \times 2FI$ is to $X\gamma q - 2AF \times X\gamma - 2BC \times 2AB$. And by multiplying the Means and Extreams into themſelves, $X\gamma\, cub. - 2AF \times X\gamma q - 4BC \times AB \times X\gamma = 8BCq \times FI$. And by adding $2AF \times X\gamma q + 4BC \times AB \times X\gamma$ to both Sides $X\gamma\, cub. = 2AF \times X\gamma q + 4BC \times AB \times X\gamma + 8BCq \times FI$. But $\frac{1}{2} X\gamma$ in the

Conſtruction to be demonſtrated was equal the Root of the Æquation $=x$, and $AF=p$, $BC=n$, $AB=\frac{q}{n}$, and $FI=\frac{r}{nn}$, and ſo $BC\times AB=q$. And $BCq\times FI=r$. Which being ſubſtituted, will make $x^3=px^2+qx+r$. Q. E. D.

*Corol.* Hence if $AF$ and $AB$ be ſuppos'd equal to nothing by the third and fourth Lemma, $2FI$ will be to $AX$ as $AX$ is to $X\gamma$, and $X\gamma$ to $2BC$. From whence ariſes the Invention of two mean Proportionals between any two given Quantities, $FI$ and $BC$.

*Scholium.* Hitherto I have only expounded the Conſtruction of a Cubick Æquation by the Ellipſe; but the Rule is of a more univerſal Nature, extending it ſelf indifferently to all the Conick Sections. For, if inſtead of the Ellipſe you would uſe the Hyperbola, take the Lines $BC$ and $BE$ on the contrary Side of the Point $B$, then let the Points $A$, $F$, $G$, $I$, $H$, $K$, $L$, and $R$ be determin'd as before, except only that $FH$ ought to be taken on the Side of $F$ not towards $I$, and that $HR$ ought to be taken in the Line $AI$ not in $HL$, on each Side the Point $H$, and inſtead of the right Line $GRS$, two other right Lines are to be drawn from the Point $L$ to the two Points $R$ and $R$ for Aſymptotes to the Hyperbola. With theſe Aſymptotes $LR$, $LR$ deſcribe an Hyperbola through the Point $G$, and a Circle from the Center $K$ with the Diſtance $GK$: And the halves of the Perpendiculars let fall from their Interſections to the right Line $AE$ will be the Roots of the Æquation propos'd. All which, the Signs + and — being rightly chang'd, are demonſtrated as above.

But if you would uſe the Parabola, the Point $E$ will be remov'd to an infinite Diſtance, and ſo not to be taken any where, and the Point $H$ will coincide with the Point $F$, and the Parabola will be to be deſcrib'd about the Axis $HL$ with the principal *Latus Rectum* $BC$ through the Points $G$ and $A$, the Vertex being plac'd on the ſame Side of the Point $F$, on which the Point $B$ is in reſpect of the Point $C$.

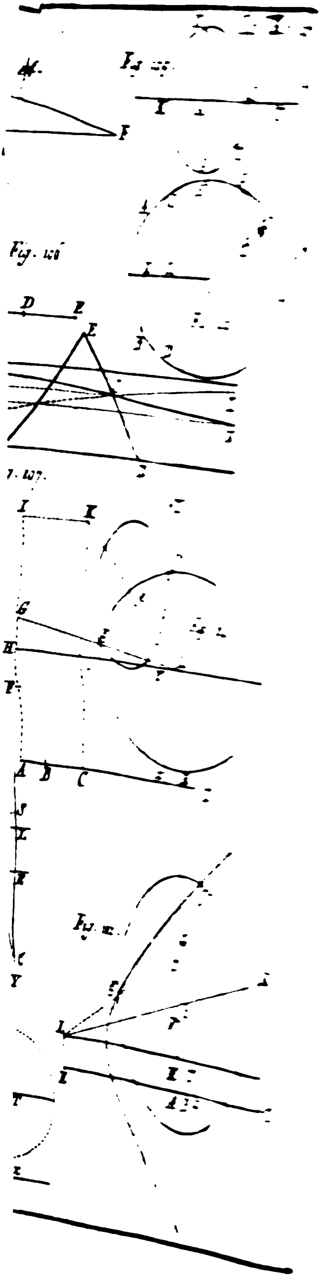
Thus the Conſtructions by the Parabola, if you regard Analytick Simplicity, are the moſt ſimple of all. Thoſe by the Hyperbola next, and thoſe which are ſolv'd by the Ellipſe,

lipſe have the third Place. But if in deſcribing of Figures, the Simplicity of the manual Operation be reſpected, the Order muſt be chang'd.

But it is to be obſerv'd in theſe Conſtructions, that by the Proportion of the principal *Latus Rectum* to the *Latus Tranſverſum*, the Species of the Ellipſe and Hyperbola may be determin'd, and that Proportion is the ſame as that of the Lines *BC* and *BE*, and therefore may be aſſum'd: But there is but one Species of the Parabola, which is obtain'd by putting *BE* infinitely long. So therefore we may conſtruct any Cubick Æquation by a Conick Section of any given Species. To change Figures given in Specie into Figures given in Magnitude, is done by encreaſing or diminiſhing all the Lines in a given Ratio, by which the Figures were given in Specie, and ſo we may conſtruct all Cubick Æquations by any given Conick Section whatever. Which is more fully explain'd thus.

Let there be propos'd any Cubick Æquation $x^3 = pxx\,.\,qx\,.\,r$, to conſtruct it by the Help of any given Conick Section. [*Vide Figures* 110 *and* 111.]

From any Point *B* in any infinite right Line *BCE*, take any two Lengths *BC*, and *BE* towards the ſame Way, if the Conick Section is an Ellipſe, but towards contrary Ways if it be an Hyperbola. But let *BC* be to *BE* as the principal *Latus-Rectum* of the given Section, is to the *Latus Tranſverſum*, and call *BC*, *n*, take $BA = \frac{q}{n}$, and that towards *C*, if *q* be Negative, and contrarily if Affirmative. At the Point *A* erect a Perpendicular *AI*, and in it take $AF = p$, and $FG = AF$; and $FI = \frac{r}{nn}$. But let *FI* be taken towards *G* if the Terms *p* and *r* have the ſame Signs, if not, towards *A*. Then make as *FH* is to *FI* ſo is *BC* to *BE*, and take this *FH* from the Point *F* towards *I*, if the Section is an Ellipſe, but towards the contrary Way if it is an Hyperbola. But let the Parallelograms *IACK* and *HAEL* be compleated, and all theſe Lines already deſcrib'd tranſferr'd to the given Conick Section; or, which is the ſame Thing, let the Curve be deſcrib'd about them, ſo that its Axis or principal tranſverſe Diameter might agree with the right Line *LA*, and the Center with the Point *L*. Theſe Things being done, let the Lines *KL* and *GL* be drawn, cutting



in *g*. In *LK* take *Lk*, which
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Tab. VIII. p. 256.

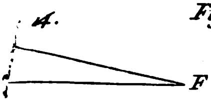
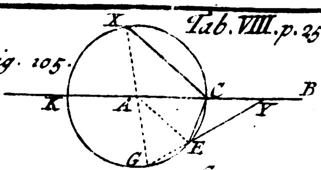


Fig. 105.



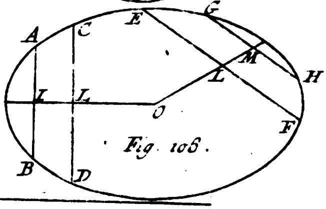
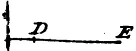
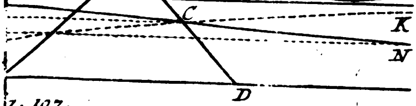


Fig. 108.

Fig. 106





y. 107.

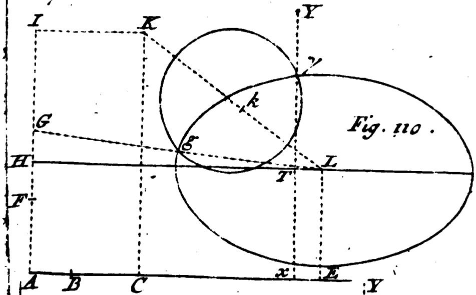
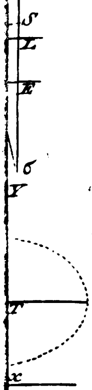


Fig. 110.



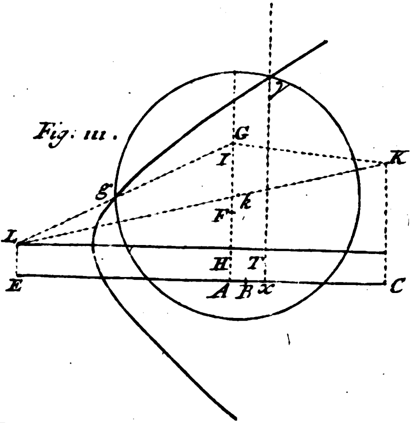


Fig. 111.

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cutting the Conick Section in $g$. In $LK$ take $Lk$, which let be to $LK$ as $Lg$ to $LG$, and from the Center $k$, with the Diſtance $kg$, deſcribe a Circle. From the Points where it cuts the given Curve, let fall Perpendiculars to the Line $LH$, whereof let $T\gamma$ be one. Laſtly, towards $\gamma$ take $TY$, which let be to $T\gamma$ as $LG$ to $Lg$, and this $TY$ produc'd will cut $AB$ in $X$, and $XY$ will be one of the Roots of the Æquation. But thoſe Roots are Affirmative which lie towards ſuch Parts of $AB$ as $FI$ lies from $F$, and thoſe are Negative which lie on the contrary Side, if $r$ is $+$; and the contrary if $r$ is $-$.

After this Manner are Cubick Æquations conſtructed by given Ellipſes and Hyperbola's: But if a Parabola ſhould be given, the Line $BC$ is to be taken equal to the Latus Rectum it ſelf. Then the Points $A$, $F$, $G$, $I$, and $K$, being found as above, a Circle muſt be deſcrib'd from the Center $K$ with the Diſtance $KG$, and the Parabola muſt be ſo apply'd to the Scheme already deſcrib'd, (or the Scheme to the Parabola) that it may paſs through the Points $A$ and $G$, and its Axis through the Point $F$ parallel to $AC$, the Vertex falling on the ſame Side of the Point $F$ as the Point $B$ falls off the Point $C$; theſe being done, if Perpendiculars were let fall from the Points where the Parabola interſects the Circle to the Line $BC$, their Halves will be equal to the Roots of the Æquation to be conſtructed.

And take Notice, that where the ſecond Term of the Æquation is wanting, and ſo the Latus Rectum of the Parabola is the Number 2, the Conſtruction comes out the ſame as that which *Des Cartes* prov'd in his Geometry, with this Difference only, that theſe Lines are the double of them.

This is a general Rule of Conſtructions. But where particular Problems are propos'd, we ought to conſult the moſt ſimple Forms of Conſtructions. For the Quantity $n$ remains free, by the taking of which the Æquation may, for the moſt part, be render'd more ſimple. One Example of which I will give.

Let there be given an Ellipſe, and let there be two mean Proportionals to be found between the given Lines $a$ and $b$. Let the firſt of them be $x$, and $a\,.\,x\,.\,\frac{xx}{a}\,.\,b$ will be continual Proportionals, and ſo $ab=\frac{x^3}{a}$, or $x^3=aab$, is the Æquation which you muſt conſtruct. Here the Terms $p$ and $q$ are wanting, and the Term $r = aab$, and therefore $BA$ and $AF$ are $= 0$, and $FI = \frac{aab}{nn}$. That the laſt Term may be more ſimple, let $n$ be aſſum'd $= a$, and let $FI = b$. And then the Conſtruction will be thus:

From any Point $A$ in any infinite right Line $AE$, take $AC = a$, and on the ſame Side of the Point $A$ take $AE$ to $AC$, as the principal Latus Rectum of the Ellipſe is to the Latus Tranſverſum. Then in the Perpendicular $AI$ take $AI = b$, and $AH$ to $AI$ as $AC$ to $AE$. [*Vide Figure* 112.] Let the Parallelograms $IACK$, $HAEL$ be compleated. Join $LA$ and $LK$. Upon this Scheme lay the given Ellipſe, and it will cut the right Line $AL$ in the Point $g$. Make $Lk$ to $LK$ as $Lg$ to $LA$. From the Center $k$, with the Diſtance $kg$, deſcribe a Circle cutting the Ellipſe in $\gamma$. Upon $AE$ let fall the Perpendicular $\gamma X$, cutting $HL$ in $T$, and let that be produc'd to $Y$, that $TY$ may be to $T\gamma$ as $TA$ to $Tg$. And ſo $XY = x$ will be equal to the firſt of the two mean Proportionals. Q. E. I.

A New

# *A New, Exact, and Eaſy Method, of finding the Roots of any Æquations Generally, and that without any previous Reduction.*

*By* Edm. Halley, *Savilian Profeſſor of Geometry.* [*Publiſh'd in the* Philoſophical Tranſactions, Numb. 210. A. D. 1694.]

THE principal Uſe of the Analytick Art, is to bring Mathematical Problems to Æquations, and to exhibit thoſe Æquations in the moſt ſimple Terms that can be. But this Art would juſtly ſeem in ſome Degree defective, and not ſufficiently Analytical, if there were not ſome Methods, by the Help of which, the Roots (be they Lines or Numbers) might be gotten from the Æquations that are found, and ſo the Problems in that reſpect be ſolved. The Antients ſcarce knew any Thing in theſe Matters beyond Quadratick Æquations. And what they writ of the Geometrick Conſtruction of ſolid Problems, by the Help of the Parabola, Ciſſoid, or any other Curve, were only particular Things deſign'd for ſome particular Caſes. But as to Numerical Extraction, there is every where a profound Silence; ſo that whatever we perform now in this Kind, is entirely owing to the Inventions of the Moderns.

And firſt of all, that great Diſcoverer and Reſtorer of the Modern Algebra, *Francis Vieta,* about 100 Years ſince, ſhew'd a general Method for extracting the Roots of any Æquation, which he publiſh'd under the Title of, *A Numerical Reſolution of Powers,* &c. *Harriot, Oughtred,* and others, as well of our own Country, as Foreigners, ought to acknowledge whatſoever they have written upon this Subject, as taken from *Vieta.* But what the Sagacity of Mr. *Newton's* Genius has perform'd in this Buſineſs we may rather conjecture (than be fully aſſur'd of) from that ſhort Specimen given

given by Dr. *Wallis* in the 94th Chapter of his *Algebra*. And we muſt be forc'd to expect it, till his great Modeſty ſhall yield to the Intreaties of his Friends, and ſuffer thoſe curious Diſcoveries to ſee the Light.

Not long ſince, (*viz. A. D.* 1690,) that excellent Perſon, Mr. *Joſeph Ralphſon*, F. R. S. publiſh'd his *Univerſal Analyſis of Æquations*, and illuſtrated his Method by Plenty of Examples; by all which he has given Indications of a Mathematical Genius, from which the greateſt Things may be expected.

By his Example, M. *de Lagney*, an ingenious Profeſſor of Mathematicks at *Paris*, was encourag'd to attempt the ſame Argument; but he being almoſt altogether taken up in extracting the Roots of pure Powers (eſpecially the Cubick) adds but little about affected Æquations, and that pretty much perplex'd too, and not ſufficiently demonſtrated: Yet he gives two very compendious Rules for the Approximation of a Cubical Root; one a Rational, and the other an Irrational one. *Ex. gr.* That the Side of the Cube $aaa+b$ is between $a+\frac{ab}{3aaa+b^{2}}$, and $\sqrt{\frac{1}{4}aa+\frac{b}{3a}}+\frac{1}{2}a$. And the Root of the 5th Power, $a^5+b$, he makes $=\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}aa}$ (where note, that 'tis $\frac{1}{4}aa$, not $\frac{1}{2}aa$, as 'tis erroneouſly printed in the French Book.) Theſe Rules were communicated to me by a Friend, I having not ſeen the Book; but having by Trial found the Goodneſs of them, and admiring the Compendium, I was willing to find out the Demonſtration. Which having done, I preſently found that the ſame Method might be accommodated to the Reſolution of all Sorts of Æquations. And I was the rather inclin'd to improve theſe Rules, becauſe I ſaw that the whole Thing might be explain'd in a *Synopſis*; and that by this means, at every repeated Step of the Calculus, the Figures already found in the Root, would be at leaſt trebled, which all other Ways are encreaſed but in an equal Number with the given ones. Now, the 'foremention'd Rules are eaſily demonſtrated from the Geneſis of the Cube, and the 5th Power. For, ſuppoſing the Side of any Cube $=a+e$, the Cube ariſing from thence is $aaa+3aae+3aee+eee$. And conſequently, if we ſuppoſe $aaa$ the next leſs Cube, to any given Non-Cubick Number, then $eee$ will be leſs than

Unity, and the Remainder $b$, will = the other Members of the Cube, $3aae+3aee+eee$. Whence rejecting $eee$ upon the Account of its Smallneſs, we have $b=3aae+3aee$. And ſince $aae$ is much greater than $aee$, the Quantity $\frac{b}{3aa}$ will not much exceed $e$; ſo that putting $e=\frac{b}{3aa}$ then the Quantity $\frac{b}{3aa+3ae}$ (to which $e$ is nearly equal) will be found $=\frac{b}{3aa+\frac{3ab}{3aa}}$, or $\frac{b}{3aa+\frac{b}{a}}$, that is, $\frac{ab}{3aa+b}=e$. And ſo the Side of the Cube $aaa+s$ will be $a+\frac{ab}{3aa+b}$, which is the *Rational Formula* of M. *de Lagney*. But now, if $aaa$ were the next greater Cubick Number to that given, the Side of the Cube $aaa-b$, will, after the ſame Manner, be found to be $a-\frac{ab}{3aa-b}$: And this eaſy and expeditious Approximation to the Cubick Root, is only (a very ſmall Matter) erroneous in point of *Defect*, the Quantity $e$, the Remainder of the Root thus found, coming ſomething leſs than really it is.

As for the *Irrational Formula*, 'tis deriv'd from the ſame Principle, *viz.* $b=3aae+3aee$, or $\frac{b}{3a}=ae+ee$, and ſo $\sqrt{\frac{1}{4}aa+\frac{b}{3a}}=\frac{1}{2}a+e$, and $\sqrt{\frac{1}{4}aa+\frac{b}{3a}}+\frac{1}{2}a=a+e$, the Root ſought. Alſo the Side of the Cube $aaa-b$, after the ſame Manner, will be found to be $\frac{1}{2}a+\sqrt{\frac{1}{4}aa-\frac{b}{3a}}$. And this *Formula* comes ſomething nearer to the Scope, being erroneous in point of *Exceſs*, as the other was in *Defect*, and is more accommodated to the Ends of Practice, ſince the Reſtitution of the Calculus is nothing elſe but the continual Addition or Subtraction of the Quantity $\frac{aee}{3a}$, according as the Quantity $e$ can be known. So that we ſhould rather write $\sqrt{\frac{1}{4}a+\frac{b-eee}{3a}}+\frac{1}{2}a$, in the former Caſe, and in the latter, $\frac{1}{2}a+\sqrt{\frac{1}{4}aa+\frac{eee-b}{3a}}$. But by either of the two *Formula's* the Figures already known in the Root to be extracted are at leaſt tripled; which I conclude will be very grateful to all the Students in Arithmetick, and I congratulate the Inventor upon the Account of his Diſcovery.

But that the Uſe of theſe Rules may be the better perceiv'd, I think it proper to ſubjoyn an Example or two. Let it be propos'd to find the Side of the double Cube, or $aaa+b=2$. Here $a=1$, and $\frac{b}{3a}=\frac{1}{2}$, and ſo $\frac{1}{2}+\sqrt{\frac{7}{12}}$, or 1,26, be found to be the true Side nearly. Now, the Cube of 1,26, is 2,000376, and ſo $0{,}63+\sqrt{{,}3969-\frac{{,}000376}{3{,}78}}$ or $0{,}63+\sqrt{{,}3968005291005291}=1{,}259921049895-$; which in 13 Figures gives the Side of the double Cube with very little Trouble, *viz.* by one only Diviſion, and the Extraction of the Square Root; when as by the common Way of working, how much Pains it would have coſt, the Skillful very well know. This Calculus a Man may continue as far as he pleaſes, by encreaſing the Square by the Addition of the Quantity $\frac{eee}{3a}$; which Correction, in this Caſe, will give but the Encreaſe of Unity in the 14th Figure of the Root.

*Example* II. Let it be propos'd to find the Sides of a Cube equal to that Engliſh Meaſure commonly call'd a Gallon, which contains 231 ſolid Ounces. The next leſs Cube is 216, whoſe Side $6=a$, and the Remainder $15=b$; and ſo for the firſt Approximation, we have $3+\sqrt{9+\frac{5}{6}}=$ the Root. And ſince $\sqrt{9{,}8333\ldots}$ is 3,1358..., 'tis plain, that $6{,}358=a+e$. Now, let $6{,}1358=a$; and we ſhall then have for its Cube 231,000853894712, and according to the Rule, $3{,}0679+\sqrt{9{,}41201041-\frac{{,}000858394712}{18{,}4070}}$ is moſt accurately equal to the Side of the given Cube, which, within the Space of an Hour, I determin'd by Calculation to be 6.13579243966195897, which is exact in the 18th Figure, defective in the 19th. And this *Formula* is deſervedly preferable to the *Rationale*, upon the Account of the great Diviſor, which is not to be manag'd without a great deal of Labour; whereas the Extraction of the Square Root proceeds much more eaſily, as manifold Experience has taught me.

But the Rule for the Root of a pure Surſolid, or the 5th Power, is of ſomething a higher Enquiry, and does much more perfectly yet do the Buſineſs; for it does at leaſt Quintuple the given Figures of the Root, neither is the Calculus very large or operoſe. Tho' the Author no where ſhews his Method of Invention, or any Demonſtration, altho' it ſeems to be very much wanting; eſpecially ſince all Things are not right in the printed Book, which may eaſily deceive the Unſkilful. Now the 5th Power of the Side $a+e$ is compos'd of theſe Members, $a^5+5a^4e+10a^3e^2+10a^2e^3+5ae^4+e^5=a^5+b$; from whence $b=5a^4e+10a^3e^2+10a^2e^3+5ae^4$, rejecting $e^5$ becauſe of its Smallneſs: Whence $\frac{b}{5a}=a^3e+2a^2e^2+2ae^3+e^4$, and adding on both Sides $\frac{1}{4}a^4$, we ſhall have $\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}=\sqrt{\frac{1}{4}a^4+a^3e+2a^2e^2+2ae^3+e^4}=\frac{1}{2}aa+ae+ee$. Then ſubtracting $\frac{1}{4}aa$ from both Sides, $\frac{1}{2}a+e$ will $=\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}aa}$; to which, if $\frac{1}{2}a$ be added, then will $a+e=\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}aa}=$ the Root of the Power $a^5+b$. But if it had $a^5-b$ (the Quantity $a$ being too great) the Rule would have been thus, $\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4-\frac{b}{5a}}-\frac{1}{4}aa}$. And this Rule approaches wonderfully, ſo that there is hardly any need of Reſtitution.

But while I conſider'd theſe Things with my ſelf, I light upon a general Method for the *Formula's* of all Powers whatſoever, and (which being handſome and conciſe enough) I thought I would not conceal from the Publick.

Theſe Formula's, (as well the *Rational* as the *Irrational* ones) are thus.

$$\sqrt{aa+b} = \sqrt{aa+b}, \text{ or } a + \frac{ab}{2aa+\frac{1}{2}b}.$$

$$\sqrt[3]{a^3+b} = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + \frac{b}{3a}}, \text{ or } a + \frac{ab}{3aaa+b}.$$

$$\sqrt[4]{a^4+b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa + \frac{b}{6aa}}, \text{ or } a + \frac{ab}{4a^4+\frac{3}{2}b}.$$

$$\sqrt[5]{a^5+b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa + \frac{b}{10a^3}}, \text{ or } a + \frac{ab}{5a^5+2b}.$$

$$\sqrt[6]{a^6+b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa + \frac{b}{15a^4}}, \text{ or } a + \frac{ab}{6a^6+\frac{5}{2}b}.$$

$$\sqrt[7]{a^7+b} = \tfrac{5}{6}a + \sqrt{\tfrac{1}{36}aa + \frac{b}{21a^5}}, \text{ or } a + \frac{ab}{7a^7+3b}.$$

And ſo alſo of the other higher Powers. But if $a$ were aſſum'd bigger than the Root ſought, (which is done with ſome Advantage, as often as the Power to be reſolv'd is much nearer, the Power of the *next greater* whole Number, than of the *next leſs*) in this Caſe, *Mutatis Mutandis*, we ſhall have the ſame Expreſſions of the Roots, *viz.*

$$\sqrt{aa-b} = \sqrt{aa-b}, \text{ or } a - \frac{ab}{2aa-\frac{1}{2}b}.$$

$$\sqrt[3]{a^3-b} = \tfrac{1}{2} + \sqrt{\tfrac{1}{4}aa - \frac{b}{3a}}, \text{ or } a - \frac{ab}{3a^3-b}.$$

$$\sqrt[4]{a^4-b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa - \frac{b}{6aa}}, \text{ or } a - \frac{ab}{4a^4-\frac{3}{2}b}.$$

$$\sqrt[5]{a^5-b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa - \frac{b}{10a^3}}, \text{ or } a - \frac{ab}{5a^5-2b}.$$

$$\sqrt[6]{a^6-b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa - \frac{b}{15a^4}}, \text{ or } a - \frac{ab}{6a^6-\frac{5}{2}b}.$$

$$\sqrt[7]{a^7-b} = \tfrac{5}{6}a + \sqrt{\tfrac{1}{36}aa - \frac{b}{21a^5}}, \text{ or } a - \frac{ab}{7a^7-3b}.$$

And within theſe two Terms the true Root is ever found; being ſomething nearer to the *Irrational* than the *Rational* Expreſſion. But the Quantity $e$ found by the *Irrational Formula*, is always too great, as the Quotient reſulting from the *Rational Formula*, is always too little. And conſequently, if we have $+b$, the *Irrational Formula* gives the Root ſomething greater than it ſhould be, and the *Rational* ſomething leſs. But contrarywiſe if it be $-b$.

And thus much may ſuffice to be ſaid concerning the Extraction of the Roots of pure Powers; which notwithſtanding, for common Uſes, may be had much more eaſily by the Help of the Logarithms. But when a Root is to be determin'd very accurately, and the Logarithmick Tables will not reach ſo far, then we muſt neceſſarily have Recourſe to theſe, or ſuch like Methods. Farther, the Invention and Contemplation of theſe *Formula*'s leading me to a certain univerſal Rule for adfected Æquations, (which I hope will be of Uſe to all the Students in *Algebra* and *Geometry*) I was willing here to give ſome Account of this Diſcovery, which I will do with all the Perſpicuity I can. I had given at N° 188. of the *Tranſactions*, a very eaſy and general Conſtruction of all adfected Æquations, not exceeding the Biquadratick Power; from which Time I had a very great Deſire of doing the ſame in Numbers. But quickly after, Mr. *Ralphſon* ſeem'd in great Meaſure to have ſatisfy'd this Deſire, till Mr. *Lagney*, by what he had perform'd in his Book, intimated, that the Thing might be done more compendiouſly yet. Now, my Method is thus:

Let $z$, the Root of any Æquation, be imagin'd to be compos'd of the Parts $a +$, or $-e$, of which, let $a$ be aſſum'd as near $z$ as is poſſible; which is notwithſtanding not *neceſſary*, but only *commodious*. Then from the Quantity $a + e$, or $a - e$, let there be form'd all the Powers of $z$, found in the Æquation, and the Numerical Co-efficients be reſpectively affix'd to them: Then let the Power to be *reſolv'd* be ſubtracted from the Sum of the given Parts (in the firſt Column where $e$ is not found) which they call the *Homogeneum Comparationis*, and let the Difference be $\pm b$. In the next Place, take the Sum of all the Co-efficients of $e$ in the ſecond Column, to which put $= s$. Laſtly, in the third Column let there be put down the Sum of all the Co-efficients of $ee$, which Sum call $t$. Then will the Root $z$ ſtand thus in the *Rational Formula*, *viz.* $z = a + \frac{sb}{ss \pm tb}$; and thus in the *Irrational Formula*, *viz.* $z = a \mp \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \mp bt}}{t}$; which perhaps it may be worth while to illuſtrate by ſome Examples. And inſtead of an *Inſtrument* let this *Table* ſerve, which ſhews the Geneſis of the ſeveral Powers of $a \pm e$, and if need be, may eaſily be continued farther; which, for its Uſe, I may rightly call a *General Analytical Speculum.* The forementſion'd Powers ariſing from a continual Multiplication by $a + e$ $(= z)$ come out thus with their adjoyn'd Co-efficients.

### *Tabula Poteſtatum.*

$$\begin{array}{llllllllllllllll}
 & & & & s & & t & & u & & w & & x & & y & \\
lz^7 & = la^7 & + & 7la^6e & + & 21la^5ee & + & 35la^4e^3 & + & 35la^3e^4 & + & 21la^2e^5 & + & 7lae^6 & + & le^7 \\
kz^6 & = ka^6 & + & 6ka^5e & + & 15ka^4ee & + & 20ka^3e^3 & + & 15ka^2e^4 & + & 6kae^5 & + & ke^6 & & \\
bz^5 & = ba^5 & + & 5ba^4e & + & 10ba^3ee & + & 10ba^2e^3 & + & 5bae^4 & + & be^5 & & & & \\
gz^4 & = ga^4 & + & 4ga^3e & + & 6ga^2ee & + & 4gae^3 & + & ge^4 & & & & & & \\
fz^3 & = fa^3 & + & 3fa^2e & + & 3faee & + & fe^3 & & & & & & & & \\
dz^2 & = da^2 & + & 2dae & + & dee & & & & & & & & & & \\
cz & = ca & + & ce & & & & & & & & & & & &
\end{array}$$

But now, if it be $a - e = z$, the Table is compos'd of the ſame Members, only the odd Powers of $e$, as $e, e^3, e^5, e^7$ are Negative, and the even Powers, as $e^2, e^4, e^6$, Affirmative. Alſo, let the Sum of the Co-efficients of the Side $e$, be $= s$; the Sum of the Co-efficients of the Square $ee = t$, the Sum of the Co-efficient of $e^3 = u$, of $e^4 = w$, of $e^5 = x$, of $e^6 = y$, &c. But now, ſince $e$ is ſuppos'd only a ſmall Part of the Root that is to be enquir'd, all the Powers of $e$ will be much leſs than the correſpondent Powers of $a$, and ſo far the firſt Hypotheſis; all the ſuperior ones may be rejected; and forming a new Æquation, by ſubſtituting $a \pm e = z$, we ſhall have (as was ſaid) $\pm b = \pm se \pm tee$. The following Examples will make this more clear.

EXAMPLE I. Let the Æquation $z^4 - 3z^2 + 75z = 10000$ be propos'd. For the firſt Hypotheſis, let $a = 10$, and ſo we have this Æquation;

$$\begin{array}{llllll}
z^4 & = +a^4 & 4a^3e & +6a^2ee & 4ae^3e & +e^4 \\
-dz^2 & = -da^2 & dae & -dee & & \\
+cz & = +ca & ce & & & \\
 & = +10000 & 4000e & +600ee & 40e^3 & +e^4 \\
 & -300 & 60e & -3ee & & \\
 & +750 & 75e & & & \\
 & -10000 & & & & \\
\hline
\end{array}$$

$$+450 - \underset{s}{4015}e + \underset{t}{597}ee - \underset{u}{40}e^3 + e^4 = 0$$

The Signs + and —, with reſpect to the Quantities $e$ and $e^3$, are left as doubtful, till it be known whether $e$ be Negative or Affirmative; which Thing creates ſome Difficulty, ſince that in Æquations that have ſeveral Roots, the *Homogenea Comparationis* (as they term them) are oftentimes encreaſed by the minute Quantity $a$, and on the contrary, *that* being encreaſed, *they* are diminiſh'd. But the Sign of $e$ is determin'd from the Sign of the Quantity $b$. For taking away the *Reſolvend* from the *Homogeneal* form'd of $a$; the Sign of $se$ (and conſequently of the prevailing Parts in the Compoſition of it) will always be contrary to the Sign of the Difference $b$. Whence 'twill be plain, whether it muſt be $+e$, or $-e$; and conſequently, whether $a$ be taken greater or leſs than the *true Root*. Now the Quantity $e$ is $= \frac{1}{2}s - \frac{\sqrt{\frac{1}{4}ss - bt}}{t}$, when $b$ and $t$ have the ſame Sign, but when the Signs are different, $e$ is $=\sqrt{\frac{\frac{1}{4}ss+bt}{t}}-\frac{1}{2}s$. But after it is found that it will be $-e$, let the Powers $e$, $e^3$, $e^5$, &c. in the affirmative Members of the Æquation be made Negative, and in the Negative be made Affirmative; that is, let them be written with the contrary Sign. On the other hand, if it be $+e$ (let thoſe forementioned Powers) be made Affirmative in the Affirmative, and Negative in the Negative Members of the Æquation.

Now we have in this Example of ours, 10450 inſtead of the Reſolvend 10000, or $b=+450$, whence it's plain, that $a$ is taken greater than the Truth, and conſequently, that 'tis $-e$. Hence the Æquation comes to be, $10450-4015e+597ee-4e^3+e^4=10000$. That is, $450-4015e+597ee=0$; and ſo $450=4015e-597ee$, or $b=se-tee$, whoſe Root $e=\frac{1}{2}s-\sqrt{\frac{\frac{1}{4}ss-bt}{t}}$, or $\frac{s}{2t}-\sqrt{\frac{ss}{4tt}-\frac{b}{t}}$; that is, in the preſent Caſe,

$e=\frac{2007\frac{1}{2}-\sqrt{3761406\frac{1}{4}}}{597}$, from whence we have the Root ſought, 9,886, which is near the Truth. But then ſubſtituting this for a ſecond Suppoſition, there comes $a+e=z$, moſt accurately, 9,886260393649 5.... ſcarce exceeding the Truth by 2 in the laſt Figure, *viz.* when $\sqrt{\frac{\frac{1}{4}ss+bt}{t}}-\frac{1}{2}s=e$. And this (if need be) may be yet much farther verify'd, by ſubtracting (if it be $+e$) the Quantity $\frac{\frac{1}{2}ne^3+\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss+tb}}$, from the Root before found; or (if it be $-e$) by adding $\frac{\frac{1}{2}ne^3-\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss-tb}}$ to that Root. Which Compendium is ſo much the more valuable, in that ſometimes from the firſt Suppoſition alone, but always from the ſecond, a Man may continue the Calculus (keeping the ſame Co-efficients) as far as he pleaſes. It may be noted, that the fore-mention'd Æquation has alſo a Negative Root *viz.* $z=10,26$.... which any one that has a Mind, may determine more accurately.

EXAMPLE II. Suppoſe $z^3 - 17z + 54z = 350$, and let $a = 10$. Then according to the Preſcript of the Rule,

$$\begin{array}{rcl} +\ z^3 & = & a^3 + 3a^2e + 3aee + e^3 \\ -dz^2 & = & da^2 - 2dae - de^2 \\ +cz & = & ca + ce \end{array}$$

$$\begin{array}{lrrrr} & & b & s & t \\ \text{That is,} & +1000 & +300e & +30e^2 & +e^3 \\ & -1700 & -340e & -17e^2 & \\ & +540 & +54e & & \\ & -350 & & & \\ \hline \text{Or,} & -510 & +14e & +13ee & +e^3 = 0 \end{array}$$

Now, ſince we have $-510$, it is plain, that $a$ is aſſumed leſs than the Truth, and conſequently that $e$ is Affirmative. And from (the Æquation) $510 = 14e + 13e^2$, comes $e = \frac{\sqrt{bt + \frac{1}{4}ss} - \frac{1}{2}s}{t} = \frac{\sqrt{6679} - 7}{13}$. Whence $z = 15,7 \ldots$; which is too much, becauſe of $a$ taken wide. Therefore, Secondly, let $a = 15$, and by the like Way of Reaſoning we ſhall find $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - tb}}{t} = \frac{109\frac{1}{2} - \sqrt{11710\frac{1}{4}}}{28}$, and conſequently, $z = 14,954068$. If the Operation were to be repeated the third Time, the Root will be found conformable to the Truth as far as the 25th Figure; but he that is contented with fewer, by writing $tb \pm te^3$ inſtead of $tb$, or ſubtracting or adding $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss \mp tb}}$ to the Root before found, will preſently obtain his End. Note, the Æquation propos'd is not explicable by any other Root, becauſe the *Reſolvend* 350 is greater than the Cube of $\frac{17}{3}$, or $\frac{d}{3}$.

EXAMPLE III. Let us take the Æquation $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$, which Dr. *Wallis* uſes *Chap.* 62. of his *Algebra*, in the Reſolution of a very difficult Arithmetical Problem, where, by *Vieta*'s Method, he has obtain'd the Root moſt accurately; and Mr. *Ralphſon* brings it alſo as an Example of his Method, *Page* 25, 26. Now this Æquation is of the Form which may have ſeveral Affirmative Roots, and (which increaſes the Difficulty) the *Co-efficients* are very great in reſpect of the *Reſolvend* given.

But that it may be the eaſier manag'd, let it be divided, and according to the known Rules of *Pointing*, let $-z^4 + 8z^3 - 20z^2 + 15z = 0,5$ (where the Quantity $z$ is $\frac{1}{10}$ of $z$ in the Æquation propos'd) and for the firſt Suppoſition, let $a = 1$. Then $+{}^{2} - 5e - 2e^2 + 4e^3 - e^4 - 0,5 = 0$; that is, $1\frac{1}{2} = 5e + 2ee$; hence $e = \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$ is $= \frac{\sqrt{37} - 5}{4}$, and ſo $z = 1,27$; whence 'tis manifeſt, that 12,7 is near the true Root of the Æquation propos'd. Now, Secondly, let us ſuppoſe $z = 12,7$, and then according to the Directions of the Table of Powers, there ariſes

$$\begin{array}{lllll}
\quad b & \quad s & \quad t & \quad u & \\
-\ 26014,4641 & -\ 8193,532e & -\ 967,74e^2 & -\ 50,8e^3 & -\ e^4 \\
+\ 153870,640 & +\ 38709,60\ e & +\ 3048\ \ e^2 & +\ 80\ \ e^3 & \\
-\ 322257,42 & -\ 50749,2\ \ e & -\ 1998\ \ e^2 & & \\
+\ 189699,9 & +\ 14937,\ \ e & & & \\
-\ 5000. & & & & \\
\hline
+\ \ 298,6559 & -\ 5296,132e & +\ 82,26e^2 & +\ 29,2e^3 & -\ e^4 = 0.
\end{array}$$

And ſo $-298,6559 = -5296,132e + 82,26ee$, whoſe Root $e$ (according to the Rule) $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, comes to $\frac{2648,066 - \sqrt{6987686,106022}}{82,26} = ,0564408033 1\ldots\ldots = e$ leſs than the Truth. But that it may be corrected, 'tis to be conſider'd, that $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - bt}}$, or $\frac{,0026201\ldots\ldots}{2643,423\ldots\ldots}$ is ,00000099, and conſequently $e$ corrected, is $= 0564470448$. And if you deſire yet more Figures of the Root, from the $e$ corrected, let there be made $tue^3 - te^4 = 0,43105602423\ldots$, and $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt - tue^3 + te^4}}{t}$, or which is all one,

$$\frac{2648,066 - \sqrt{6987685,67496597577\ldots}}{82,26} =$$

$,05644179448074402 = e$; whence $a + e = z$ the Root is moſt accurately $12,75644179448074402\ldots$, as Dr. *Wallis* found in the forementioned Place; where it may be obſerv'd, that the Repetition of the *Calculus* docs ever triple the true Figures in the aſſum'd $a$, which the firſt Correction, or

$\frac{1}{2}ue$

$\frac{\frac{1}{2}ne^3-\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss-bt}}$ does quintuple; which is alſo commodiouſly done by the *Logarithms*. But the other Correction after the firſt, does alſo double the Number of Figures, ſo that it renders the *aſſumed* altogether Seven-fold; yet the firſt Correction is abundantly ſufficient for Arithmetical Uſes, for the moſt Part.

But as to what is ſaid concerning the Number of Places rightly taken in the Root, I would have underſtood ſo, that when $a$ is but $\frac{1}{10}$ Part diſtant from the true Root, then the firſt Figure is rightly aſſumed; if it be within $\frac{1}{100}$ Part, then the two firſt Figures are rightly aſſumed; if within $\frac{1}{1000}$, and then the three firſt are ſo; which conſequently, manag'd according to our Rule, do preſently become nine Figures.

It remains now that I add ſomething concerning our *Rational Formula*, viz. $e=\frac{sb}{ss\pm tb}$, which ſeems expeditious enough, and is not much inferior to the former, ſince it will triple the given Number of Places. Now, having form'd an Æquation from $a\pm e=z$, as before, it will preſently appear, whether $a$ be taken greater or leſſer than the Truth; ſince $se$ ought always to have a Sign contrary to the Sign of the Difference of the *Reſolvend*, and its *Homogeneal* produc'd from $a$. Then ſuppoſing $+b+se+a-tee=0$, the Diviſor is $ss-tb$, as often as $t$ and $b$ have the ſame Signs; but it is $ss+bt$, when they have different ones. But it ſeems moſt commodious for Practice, to write the *Theorem* thus, $e=\frac{b}{s}+\frac{tb}{s}$, ſince this Way the Thing is done by one Multiplication and two Diviſions, which otherwiſe would require three Multiplications, and one Diviſion.

Let us take now one Example of this Method, from the Root (of the foremention'd Æquation) 12,7 ...., where

$$\begin{array}{lllll} 298{,}6559 & -5296{,}132e & +82{,}26ee & +29{,}2e^3 & -e^4=0, \\ +b & -s & +t & +n & \end{array}$$

and ſo $\frac{b}{s}-\frac{tb}{s}=e$; that is, let it be as $s$ to $t$, ſo $b$ to $\frac{tb}{s}=5296{,}132$) 298,6559 into 82,26 (4,63875... wherefore the Diviſor is $s-\frac{tb}{s}=5291{,}49325$ .....) 298,6559

(0,056441

(0,056441 .... = $e$, that is, to five true Figures, added to the Root that was taken. But this *Formula* cannot be corrected, as the foregoing *Irrational* one was; and ſo if more Figures of the Root are deſired, 'tis the beſt to make a new Suppoſition, and repeat the *Calculus* again: And then a new Quotient, tripling the known Figures of the Root, will abundantly ſatisfy even the moſt Scrupulous.

# *FINIS.*

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